

# LOCAL WELL-POSEDNESS FOR THE DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION IN BESOV SPACES

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**ABSTRACT.** It is shown that the cubic derivative nonlinear Schrödinger equation is locally well-posed in Besov spaces  $B_{2,\infty}^s(\mathbb{X})$ ,  $s \geq \frac{1}{2}$ , where we treat the non-periodic setting  $\mathbb{X} = \mathbb{R}$  and the periodic setting  $\mathbb{X} = \mathbb{T}$  simultaneously. The proof is based on the strategy of Herr for initial data in  $H^s(\mathbb{T})$ ,  $s \geq \frac{1}{2}$ .

## 1. INTRODUCTION AND MAIN RESULT

We study the Cauchy problem for the following derivative nonlinear Schrödinger equation:

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\partial_x(|u|^2 u) + \lambda|u|^{2k}u & \text{in } \mathbb{X} \times (-T, T), \\ u(0) = u_0 & \text{in } \mathbb{X}, \end{cases} \quad (1)$$

where  $\lambda \in \mathbb{R}$ ,  $k \in \mathbb{N}_0$ ,  $T > 0$ ,  $\mathbb{X} = \mathbb{R}$  (*non-periodic setting*) or  $\mathbb{X} = \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$  (*periodic setting*). We look for solutions  $u$  which satisfy the corresponding integral equation

$$u(t) = U_t u_0 + \int_0^t U_{t-t'} [\partial_x(|u|^2 u)(t') - i\lambda|u|^{2k}u(t')] dt', \quad t \in (-T, T),$$

where  $(U_t u_0)^\wedge(\xi) = e^{-it\xi^2} \widehat{u_0}(\xi)$  for  $u_0 \in \mathcal{S}(\mathbb{X})$ .

In the non-periodic setting, Takaoka [17] showed local well-posedness for initial data  $u_0 \in H^s(\mathbb{R})$  and  $s \geq \frac{1}{2}$  which improved the results of Hayashi and Ozawa [11, 10, 12] for initial data in  $H^1(\mathbb{R})$ . The central tools have been Fourier restriction methods, local smoothing, Strichartz estimates and a gauge transformation which cancels out the unfavorable nonlinear term  $2i|u|^2 \partial_x u$ .

In the periodic setting, Herr [13] showed local well-posedness in  $H^s(\mathbb{T})$ ,  $s \geq \frac{1}{2}$ , by using an adapted gauge transformation and a suitable version of Bourgain's  $L^4$ -Strichartz estimate.

For  $s < \frac{1}{2}$ , Biagioni and Linares [1] showed that the flow map  $u_0 \mapsto u$  is no longer uniformly continuous. In this sense,  $H^{1/2}$  is critical. However, with respect to scaling,  $L^2$  is critical: If  $u$  solves (1) with initial datum  $u_0$ , then  $u_\sigma(x, t) := \frac{1}{\sigma^{1/2}} u\left(\frac{x}{\sigma}, \frac{t}{\sigma^2}\right)$ ,  $\sigma > 0$ , is a solution for initial data  $u_0(\frac{\cdot}{\sigma})$  and we have  $\|u_\sigma(t)\|_{L^2} = \|u(\frac{t}{\sigma^2})\|_{L^2}$ . In

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order to meet this gap between  $L^2$  and  $H^{1/2}$ , Grünrock and Herr [6, 7] proved local well-posedness in spaces  $\widehat{H}_p^s(\mathbb{X})$ , where

$$\|f\|_{\widehat{H}_p^s} := \|\widehat{J^s f}\|_{L^{p'}}$$

for  $s \geq \frac{1}{2}$  and  $1 < p \leq 2$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$ . In the non-periodic setting, S. Guo, Ren and Wang [8] recently generalized this result to modulation spaces  $M_{2,q}^s$  with

$$\|f\|_{M_{2,q}^s} := \left( \sum_{k \in \mathbb{Z}} \langle k \rangle^{sq} \|\mathbb{1}_{[k-\frac{1}{2}, k+\frac{1}{2}]} \widehat{f}\|_{L^2}^q \right)^{1/q}$$

for  $q \in [2, \infty)$  and  $s \geq \frac{1}{2}$ . In the scaling sense,  $M_{2,q}^{1/2}$  is subcritical for  $2 \leq q < \infty$  and critical for  $q = \infty$ .

We show local well-posedness for initial data in  $B_{2,\infty}^s(\mathbb{X})$ ,  $s \geq \frac{1}{2}$ :

**Theorem 1.1.** *Let  $s \geq \frac{1}{2}$  and  $k \in \mathbb{N}_0$ . For any  $r > 0$ , there exists  $T = T(r) > 0$  and a metric space  $M_{s,T}$ , such that for all  $u_0 \in B_r := \{f \in B_{2,\infty}^s : \|f\|_{B_{2,\infty}^s} < r\}$ , the equation (1) has a unique solution  $u \in M_{s,T} \hookrightarrow \mathcal{C}([-T, T], B_{2,\infty}^s)$ . The flow map*

$$\tilde{F}: B_r \rightarrow \mathcal{C}([-T, T], B_{2,\infty}^s), \quad u_0 \mapsto u$$

*is continuous.*

Therefore, we point out that the method of [13] is also applicable to the non-periodic setting with some slight modifications. We extend this method to the Besov space setting by using several frequency-localization arguments. Noticing that  $H^{1/2}(\mathbb{X}) \hookrightarrow B_{2,\infty}^{1/2}(\mathbb{X})$ , we improve the results of Takaoka [17] and Herr [13].

Global well-posedness was shown by Hayashi and Ozawa [11, 10, 12] in the non-periodic setting for  $u_0 \in H^1$  with mass  $\|u_0\|_{L^2}^2 < 2\pi$ . For  $\lambda = 0$ , Z. Guo and Wu [18, 9] generalized this result to  $u_0 \in H^{1/2}$  with mass smaller than  $4\pi$ . Recently, Mosincat [16] proved the same result in the periodic setting.

There are also results for global weak solutions in Sobolev spaces corresponding to  $H^1$  concerning Dirichlet and generalized periodic boundary conditions, compare for example [2, 15].

The remainder of this paper is organized as follows: We complete this section with some general notation. In the second section, we briefly introduce the Gauge transformation and the Gauge equivalent Cauchy problem. In the third section, we establish the function spaces and basic estimates for the linear and the Duhamel term. In the fourth section, we prove the estimate for the trilinear derivative term  $u^2 \partial_x \overline{u}$ . The fifth section treats the multilinear terms  $|u|^{2k} u$  and  $|u|^4 u$ . In the last section, we conclude local well-posedness for the Gauge equivalent problem which implies the statement of theorem 1.1 by backward transformation.

**Notation.** For  $a, b \geq 0$ , we denote  $a \lesssim b$  if  $a \leq cb$  for some  $c > 0$ ,  $a \ll b$  if  $Ca < b$  for a sufficiently large  $C > 1$  and  $a \sim b$  if  $C^{-1}a \leq b \leq Ca$  for a sufficiently large  $C > 1$ . We write  $\lesssim_\alpha$  if the implicit constants depends on a parameter  $\alpha$ .

For measure spaces  $\Omega_1, \Omega_2$  and product-measurable mappings  $u: \Omega_1 \times \Omega_2 \rightarrow \mathbb{C}$ ,  $(x, t) \mapsto u(x, t)$  such that  $u(\cdot, t) \in X$  and  $u(x, \cdot) \in Y$ , we set

$$\|u\|_{Y_t X_x} := \|t \mapsto \|x \mapsto u(x, t)\|_X\|_Y$$

and shortly  $\|u\|_{X_{t,x}} := \|u\|_{X_t X_x}$  if  $X = Y$ .

$\mathcal{S}(\mathbb{R}^n)$  denotes the space of all *Schwartz functions* on  $\mathbb{R}^n$ ,  $\mathcal{S}(\mathbb{T} \times \mathbb{R})$  the space of all functions  $u: \mathbb{R}^2 \rightarrow \mathbb{C}$  such that

$$u(x + 2\pi, t) = u(x, t), \quad u(\cdot, t) \in \mathcal{C}^\infty(\mathbb{R}), \quad u(x, \cdot) \in \mathcal{S}(\mathbb{R})$$

and  $\mathcal{S}(\mathbb{T})$  stands for the space of  $2\pi$ -periodic  $\mathcal{C}^\infty$ -functions on  $\mathbb{R}$ .

For  $f \in \mathcal{S}(\mathbb{R})$ , we define the *Fourier transform*  $\widehat{f}$  via

$$\widehat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix \cdot \xi} f(x) dx, \quad \xi \in \mathbb{R}.$$

For  $f \in L^1(\mathbb{T})$  and  $g \in L^1(\mathbb{Z})$ , we denote

$$\widehat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} e^{-ix \cdot \xi} f(x) dx, \quad \xi \in \mathbb{Z}.$$

For  $u \in (\mathbb{X} \times \mathbb{R})$ , we set

$$\widehat{u}(\xi, \tau) := \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{X}} e^{i(x,t) \cdot (\xi, \tau)} u(x, t) dx dt.$$

$J^s$  denotes the Bessel potential of order  $-s$ . This means

$$\widehat{J^s f} = \langle \cdot \rangle^s \widehat{f}, \quad f \in \mathcal{S}(\mathbb{X}),$$

where  $\langle a \rangle := (1 + a^2)^{1/2}$ ,  $a \in \mathbb{R}$ .

$H^s(\mathbb{X})$  is the *Sobolev space* of order  $s$  on  $\mathbb{X}$ , this means the completion of  $\mathcal{S}(\mathbb{X})$  with respect to the norm

$$\|f\|_{H^s} := \|J^s f\|_{L^2}.$$

For  $u \in \mathcal{S}(\mathbb{X} \times \mathbb{R})$  and  $s \in \mathbb{R}$ , we write

$$\begin{aligned} J_x^s u(x, t) &:= J^s(u(t))(x), \\ \widehat{\Gamma_\pm^s u}(\xi, \tau) &:= \langle \tau \pm \xi^2 \rangle^s \widehat{u}(\xi, \tau) \end{aligned}$$

and shortly  $\Gamma := \Gamma_+$ .

We consider  $\vec{\xi} = (\xi_1, \dots, \xi_n)$  and  $\mathbb{Y}_\xi^n := \{\vec{\xi} \in \mathbb{Y}^n : \sum_{j=1}^n \xi_j = \xi\}$ ,  $\mathbb{Y} \in \{\mathbb{R}, \mathbb{Z}\}$ . The convolution of functions  $f_1, \dots, f_n$  on  $\mathbb{Y}$  is written as

$$\begin{aligned} f_1 * \dots * f_n(\xi) &= \int_{\mathbb{Y}^{n-1}} \prod_{j=1}^{n-1} f_j(\xi_j) \cdot f_n(\xi - \xi_1 - \dots - \xi_{n-1}) d(\xi_1, \dots, \xi_{n-1}) \\ &=: \int_{\mathbb{Y}_\xi^n} \prod_{j=1}^n f_j(\xi_j) d\vec{\xi}, \end{aligned}$$

with integration with respect to the counting measure if  $\mathbb{Y} = \mathbb{Z}$ .

Let  $\chi \in \mathcal{C}^\infty(\mathbb{R}, [0, 1])$  be radially decreasing such that  $\chi = 1$  on  $[-1, 1]$  and  $\chi = 0$  on  $(-2, 2)^c$ . For  $T > 0$ , we introduce

$$\chi_T(\xi) := \chi\left(\frac{\xi}{T}\right) - \chi\left(\frac{2\xi}{T}\right), \quad \chi_{\leq T}(\xi) := \chi\left(\frac{\xi}{T}\right).$$

Hence  $\text{supp } \chi_T \subseteq \{\xi \in \mathbb{R} : \frac{T}{2} < |\xi| < 2T\}$ . In addition, let

$$\mathcal{D} := \{2^n : n \in \mathbb{Z}\} = \{N : N \text{ dyadic}\}, \quad \mathcal{D}_1 := \{N \geq 1 : N \in \mathcal{D}\}.$$

For  $\xi \neq 0$ , there are not more than two  $N \in \mathcal{D}$  such that  $\chi_N(\xi) \neq 0$ . We have  $\sum_{N \in \mathcal{D}} \chi_N(\xi) = 1$  for all  $\xi \neq 0$  and  $\chi_{\leq 1}(\xi) + \sum_{N \in \mathcal{D}} \chi_N(\xi) = 1$  for all  $\xi \in \mathbb{R}$ . For  $N \in \mathcal{D}_1$ ,  $f \in \mathcal{S}(\mathbb{X})$  and  $u \in \mathcal{S}(\mathbb{R} \times \mathbb{X})$ , we denote

$$\widehat{P_N f}(\xi) := \begin{cases} \chi_N(\xi) \widehat{f}(\xi) & \text{for } N > 1, \\ \chi_{\leq 1}(\xi) \widehat{f}(\xi) & \text{for } N = 1, \end{cases}$$

$$P_N u(x, t) := P_N(u(t))(x).$$

This means  $\sum_{N \in \mathcal{D}_1} P_N f = f$ .

Finally, let  $\chi_{[0,1]} \in \mathcal{C}^\infty(\mathbb{R})$  be a radially decreasing function satisfying  $\chi_{[0,1]} = 1$  on  $[0, 1]$  and  $\chi_{[0,1]} = 0$  on  $(-1, 2)^c$ . For intervals  $[a, b]$ , we denote  $\chi_{[a,b]}(\xi) = \chi_{[0,1]}(\frac{\xi-a}{b-a})$  and

$$P_{[a,b]} f(\xi) := \chi_{[a,b]}(\xi) \widehat{f}(\xi), \quad P_{[a,b]} u(x, t) := P_{[a,b]}(u(t))(x).$$

## 2. GAUGE TRANSFORMATION

We work with the gauge transformation as introduced by Hayashi and Ozawa [11] for the non-periodic setting and adapted by Herr [13] for the periodic setting.

**Definition 2.1: Gauge transformation.** For

$$J(f)(x) := \int_{-\infty}^x |f(y)|^2 dy,$$

$$\mathcal{J}(f)(x) := \frac{1}{2\pi} \int_0^{2\pi} \int_{\vartheta}^x (|f(y)|^2 - \mu(f)) dy d\vartheta, \quad \mu(f) := \frac{1}{2\pi} \|f\|_{L^2(\mathbb{T})}^2,$$

we define

$$G(f)(x) := e^{-iJ(f)(x)} f(x), \quad f \in L^2(\mathbb{R}),$$

$$\mathcal{G}(f)(x) := e^{-i\mathcal{J}(f)(x)} f(x), \quad f \in L^2(\mathbb{T}),$$

$$G(u)(x, t) := G(u(t))(x), \quad u \in \mathcal{C}([-T, T], L^2(\mathbb{R})),$$

$$\mathcal{G}(u)(x, t) := \mathcal{G}(u(t))(x - 2\mu(u)t), \quad u \in \mathcal{C}([-T, T], L^2(\mathbb{T})).$$

As shown in [12] and [13], we can consider the gauge equivalent problems

$$\begin{cases} i\partial_t v + \partial_x^2 v = -iv^2 \partial_x \bar{v} - \frac{1}{2}|v|^4 v + \lambda|v|^{2k} v & \text{in } \mathbb{R} \times (-T, T), \\ v(0) = v_0 & \text{in } \mathbb{R}, \end{cases}$$

and

$$\begin{cases} i\partial_t v + \partial_x^2 v = -iv^2 \partial_x \bar{v} - \frac{1}{2}|v|^4 v + \lambda|v|^{2k} v + \mu(v)|v|^2 v - \psi(v)v & \text{in } \mathbb{T} \times (-T, T), \\ v(0) = v_0 & \text{in } \mathbb{T}, \end{cases}$$

where  $\mu$  is defined as above and  $\psi(v)(t) := \frac{1}{2\pi} \int_0^{2\pi} (2\text{Im}(v\partial_x \bar{v})(y, t) - \frac{1}{2}|v|^4(y, t)) dy + \mu(v)^2$ . Denoting

$$\mathcal{T}(v)(t) := v(t)^2 \partial_x \bar{v}(t), \quad \mathcal{Q}(v)(t) := |v(t)|^4 v(t)$$

in the non-periodic setting and

$$\begin{aligned} \mathcal{T}(v)(t) &:= v(t)^2 \partial_x \bar{v}(t) - \frac{i}{2\pi} v(t) \int_0^{2\pi} 2\text{Im}(v\partial_x \bar{v})(y, t) dy, \\ \mathcal{Q}(v)(t) &:= \left( |v(t)|^4 - \frac{1}{2\pi} \int_0^{2\pi} |v(t)|^4 dx \right) v(t) \\ &\quad - \frac{1}{\pi} \int_0^{2\pi} |v(t)|^2 dx \left( |v(t)|^2 - \frac{1}{2\pi} \int_0^{2\pi} |v(t)|^2 dx \right) v(t) \end{aligned}$$

in the periodic setting leads to the Cauchy problem

$$\begin{cases} i\partial_t v + \partial_x^2 v = -i\mathcal{T}(v) - \frac{1}{2}\mathcal{Q}(v) + \lambda|v|^{2k} v & \text{in } \mathbb{X} \times (-T, T), \\ v(0) = v_0 & \text{in } \mathbb{X}. \end{cases} \quad (2)$$

In order to treat both settings simultaneously, the following characterizations of  $\mathcal{T}$  and  $\mathcal{Q}$  are helpful:

**Lemma 2.2.** *We have  $\mathcal{T}(v) = \mathcal{T}(v, v, \bar{v})$  and  $\mathcal{Q}(v) = \mathcal{Q}(v, \bar{v}, v, \bar{v}, v)$ , where*

$$\begin{aligned} \mathcal{T}(v_1, v_2, v_3)^\wedge(\xi, \tau) &= \frac{1}{2\pi} \int_{\mathbb{R}_\tau^3} \int_{\mathbb{R}_\xi^3} \widehat{v}_1(\xi_1, \tau_1) \widehat{v}_2(\xi_2, \tau_2) i\xi_3 \widehat{v}_3(\xi_3, \tau_3) d\vec{\xi} d\vec{\tau}, \\ \mathcal{Q}(v_1, v_2, v_3, v_4, v_5)^\wedge(\xi, \tau) &= \frac{1}{2\pi} \int_{\mathbb{R}_\tau^3} \int_{\mathbb{R}_\xi^3} \prod_{j=1}^5 \widehat{v}_j(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \end{aligned}$$

in the non-periodic setting and

$$\begin{aligned} \mathcal{T}(v_1, v_2, v_3)^\wedge(\xi, \tau) &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}_\tau^3} \sum_{\substack{\xi_1 + \xi_2 + \xi_3 = \xi \\ \xi_1, \xi_2 \neq \xi}} \widehat{v}_1(\xi_1, \tau_1) \widehat{v}_2(\xi_2, \tau_2) i\xi_3 \widehat{v}_3(\xi_3, \tau_3) d\vec{\tau} \\ &\quad + \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}_\tau^3} \widehat{v}_1(\xi, \tau_1) \widehat{v}_2(\xi, \tau_2) i\xi \widehat{v}_3(-\xi, \tau_3) d\vec{\tau}, \\ \mathcal{Q}(v_1, v_2, v_3, v_4, v_5)^\wedge(\xi, \tau) &= \frac{1}{(2\pi)^{5/2}} \int_{\mathbb{R}_\tau^3} \sum_{\substack{\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 = \xi \\ \xi_1 + \xi_2 + \xi_3 + \xi_4, \xi_1 + \xi_2, \xi_3 + \xi_4 \neq 0}} \prod_{j=1}^5 \widehat{v}_j(\xi_j, \tau_j) d\vec{\tau} \end{aligned}$$

in the periodic setting.

*Proof.* In the non-periodic case, this is a direct consequence of the elementary properties of the Fourier transformation. For the periodic setting, compare [7, Lemma 6.4].  $\square$

## 3. SPACES: DEFINITION AND BASIC PROPERTIES

**Definition 3.1.** For  $s \in \mathbb{R}$ ,  $p \in (0, \infty]$ ,  $q \in (0, \infty]$ , we define the *Besov space*  $B_{p,q}^s(\mathbb{X})$  as the completion of  $\mathcal{S}(\mathbb{X})$  with respect to the norm

$$\|f\|_{B_{p,q}^s(\mathbb{X})} := \begin{cases} \|P_1 f\|_{L^p(\mathbb{X})} + \left( \sum_{N>1} N^{sq} \|P_N f\|_{L^p(\mathbb{X})}^q \right)^{1/q} & \text{if } q < \infty, \\ \|P_1 f\|_{L^p(\mathbb{X})} + \sup_{N>1} N^s \|P_N f\|_{L^p(\mathbb{X})} & \text{if } q = \infty, \end{cases}$$

where we take the supremum and the sum over dyadic numbers  $N$ .

**Definition 3.2.** Let  $s, b \in \mathbb{R}$ . We define  $X^{s,b,\pm}(\mathbb{X})$ ,  $Y^{s,b}(\mathbb{X})$ ,  $Z^s(\mathbb{X})$ ,  $\mathfrak{X}^{s,b,\pm}(\mathbb{X})$ ,  $\mathcal{Y}^{s,b}(\mathbb{X})$ ,  $\mathcal{Z}^s(\mathbb{X})$  as the completions of  $\mathcal{S}(\mathbb{X} \times \mathbb{R})$  with respect to the norms

$$\begin{aligned} \|u\|_{X^{s,b,\pm}} &:= \|\langle \xi \rangle^s \langle \tau \pm \xi^2 \rangle^b \widehat{u}(\xi, \tau)\|_{L_\xi^2(\mathbb{Y}) L_\tau^2(\mathbb{R})}, \\ \|u\|_{Y^{s,b}} &:= \|\langle \xi \rangle^s \langle \tau + \xi^2 \rangle^b \widehat{u}(\xi, \tau)\|_{L_\xi^2(\mathbb{Y}) L_\tau^1(\mathbb{R})}, \\ \|u\|_{Z^s} &:= \|u\|_{X^{s,\frac{1}{2}}} + \|u\|_{Y^{s,0}}, \\ \|u\|_{\mathfrak{X}^{s,b,\pm}} &:= \|P_1 u\|_{X^{s,b,\pm}} + \sup_{N>1} \|P_N u\|_{X^{s,b,\pm}}, \\ \|u\|_{\mathcal{Y}^{s,b}} &:= \|P_1 u\|_{Y^{s,b}} + \sup_{N>1} \|P_N u\|_{Y^{s,b}}, \\ \|u\|_{\mathcal{Z}^s} &:= \|P_1 u\|_{Z^s} + \sup_{N>1} \|P_N u\|_{Z^s}. \end{aligned}$$

For  $T > 0$ , we consider the space  $\mathcal{Z}_T^s(\mathbb{X}) := \left\{ u|_{[-T,T]} : u \in \mathcal{Z}^s(\mathbb{X}) \right\}$  with norm

$$\|u\|_{\mathcal{Z}_T^s} := \inf \left\{ \|v\|_{Z^s} : u = v|_{[-T,T]}, v \in \mathcal{Z}^s(\mathbb{X}) \right\}.$$

The following estimates for the linear term, the Duhamel term and for the behavior under multiplication with smooth cutoffs can be found e.g. in [5, 13] for the case without frequency-localization. With trivial modifications, they remain true in our setting:

**Lemma 3.3.** Let  $s \geq 0$ ,  $u_0 \in B_{2,\infty}^s(\mathbb{X})$  and  $w \in \mathcal{S}(\mathbb{X} \times \mathbb{R})$  such that  $\text{supp } w \subseteq \mathbb{X} \times [-2, 2]$ . Then

$$\begin{aligned} \|\chi(t) U_t u_0\|_{Z^s} &\lesssim \|u_0\|_{B_{2,\infty}^s}, \\ \left\| \chi(t) \int_0^t U_{t-t'} w(t') dt' \right\|_{Z^s} &\lesssim \|w\|_{\mathfrak{X}^{s,-\frac{1}{2}} \cap \mathcal{Y}^{s,-1}}. \end{aligned}$$

For  $u \in \mathcal{S}(\mathbb{X} \times \mathbb{R})$ ,  $s \in \mathbb{R}$ ,  $0 \leq b_1 < b_2 < \frac{1}{2}$ ,  $N \in \mathcal{D}_1$ ,  $\delta > 0$  and  $T \in (0, 1]$ , we have

$$\begin{aligned} \|P_N(\chi_T(t)u)\|_{Y^{s,0}} &\lesssim \|P_N u\|_{Y^{s,0}}, \\ \|P_N(\chi_T(t)u)\|_{X^{s,\frac{1}{2}}} &\lesssim T^{-\delta} \|P_N u\|_{X^{s,\frac{1}{2}}}, \\ \|P_N(\chi_T(t)u)\|_{X^{s,b_1,\pm}} &\lesssim T^{b_2-b_1} \|P_N u\|_{X^{s,b_2,\pm}}. \end{aligned}$$

For  $T > 0$ , the embedding  $\mathcal{Z}_T^s \hookrightarrow \mathcal{C}([-T, T], B_{2,\infty}^s)$  holds true.

The following statements can be found again in [5, 13]:

**Lemma 3.4.** *Let  $u \in \mathcal{S}(\mathbb{X} \times \mathbb{R})$ . Then*

$$\|u\|_{Y^{s,b_1}} \lesssim \|u\|_{X^{s,b_2}} \quad \forall b_2 > b_1 + \frac{1}{2}, \quad (3)$$

$$\|u\|_{L_t^p L_x^q} \lesssim \|u\|_{X^{s,b,\pm}} \quad \forall p, q \in [2, \infty), \quad b \geq \frac{1}{2} - \frac{1}{p}, \quad s \geq \frac{1}{2} - \frac{1}{q}, \quad (4)$$

$$\|u\|_{X^{s,b,\pm}} \lesssim \|u\|_{L_t^p H_x^s} \quad \forall p \in (1, 2], \quad b \leq \frac{1}{2} - \frac{1}{p}. \quad (5)$$

**Lemma 3.5: Strichartz Estimates.** *Let  $u \in \mathcal{S}(\mathbb{X} \times \mathbb{R})$ . We have*

$$\|u\|_{L_{t,x}^4} \lesssim \|u\|_{X^{0,b,\pm}} \quad \forall b > \frac{3}{8}, \quad (6)$$

$$\|u\|_{X^{0,b,\pm}} \lesssim \|u\|_{L_{t,x}^{4/3}} \quad \forall b < -\frac{3}{8}. \quad (7)$$

For  $\mathbb{X} = \mathbb{R}$  and  $b > \frac{1}{2}$ ,  $p \in (2, \infty]$ ,  $q \in [2, \infty]$  satisfying  $\frac{2}{p} + \frac{1}{q} = \frac{1}{2}$ , it also holds that

$$\|u\|_{L_t^p L_x^q} \lesssim \|u\|_{X^{0,b,\pm}}. \quad (8)$$

Finally, for  $\tilde{p}$  with  $\frac{1}{\tilde{p}} = \vartheta \frac{1}{6} + (1 - \vartheta) \frac{1}{2}$ ,  $\vartheta \in (0, 1)$ ,  $b > \frac{1}{2}$ , we have

$$\|u\|_{L_{t,x}^{\tilde{p}}} \lesssim \|u\|_{X^{0,\vartheta b,\pm}}. \quad (9)$$

*Proof.* Since  $\|\cdot\|_{L_t^p L_x^q}$  is invariant under complex conjugation, it suffices to consider  $\|\cdot\|_{X^{s,b,+}}$ . In the periodic setting, (6) and (7) have been shown in [4, Lemma 2.1]. Since  $(p, q)$  is a Strichartz pair, we obtain (8), compare for example [3, Lemma 2.3]. Estimate (9) can be concluded by interpolation (compare for example [5, Lemma 1.4]) between (8) and the trivial statement

$$\|u\|_{L_{t,x}^2} = \|u\|_{X^{0,0}}.$$

In the non-periodic setting, (6) is a direct consequence of (9) by plugging in  $\vartheta = \frac{3}{4}$ . Finally, estimate (7) follows from (6) and duality.  $\square$

#### 4. TRILINEAR ESTIMATE

In this section, we handle the trilinear term  $\mathcal{T}(u)$  which is essentially  $u^2 \partial_x \bar{u}$ . The partial derivative on the third factor causes a factor  $\xi_3$  on the Fourier side. If  $|\xi_3|$  is significantly higher than the first two frequencies, we can use the following *resonance relation* to control the derivative term: For  $(\xi, \tau) \in \mathbb{R}^2$ ,  $(\xi_1, \xi_2, \xi_3), (\tau_1, \tau_2, \tau_3) \in \mathbb{R}^3$  such that  $\xi_1 + \xi_2 + \xi_3 = \xi$  and  $\tau_1 + \tau_2 + \tau_3 = \tau$ , we have

$$\begin{aligned} & 4 \max\{|\tau + \xi^2|, |\tau_1 + \xi_1^2|, |\tau_2 + \xi_2^2|, |\tau_3 - \xi_3^2|\} \\ & \geq |\tau + \xi^2 - (\tau_1 + \xi_1^2 + \tau_2 + \xi_2^2 + \tau_3 - \xi_3^2)| \\ & = 2|\xi_1 + \xi_3||\xi_2 + \xi_3|. \end{aligned} \quad (10)$$

For  $|\xi_3| \gg |\xi_1|, |\xi_2|$ , we can conclude  $\max\{|\tau + \xi^2|, |\tau_1 + \xi_1^2|, |\tau_2 + \xi_2^2|, |\tau_3 - \xi_3^2|\} \gtrsim \xi_3^2$ .

In the sequel, we consider the multipliers of [13] with some slight modifications:

**Definition 4.1.** Let  $j \in \{1, 2, 3\}$ ,  $\xi \in \mathbb{Y}$ ,  $\tau \in \mathbb{R}$ ,  $\xi_j \in \mathbb{Y}$ ,  $\tau_j \in \mathbb{R}$ ,  $\vec{\xi} = (\xi_1, \xi_2, \xi_3)$ ,  $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ . We set  $A(\xi, \tau, \vec{\xi}, \vec{\tau}) := \{|\tau + \xi^2|, |\tau_1 + \xi_1^2|, |\tau_2 + \xi_2^2|, |\tau_3 - \xi_3^2|\}$ ,

$$\begin{aligned} A_0(\xi, \tau) &:= \{(\vec{\xi}, \vec{\tau}) \in \mathbb{Y}_\xi^3 \times \mathbb{R}_\tau^3 : \max A(\xi, \tau, \vec{\xi}, \vec{\tau}) = |\tau + \xi^2|\}, \\ A_1(\xi, \tau) &:= \{(\vec{\xi}, \vec{\tau}) \in \mathbb{Y}_\xi^3 \times \mathbb{R}_\tau^3 : \max A(\xi, \tau, \vec{\xi}, \vec{\tau}) = |\tau_1 + \xi_1^2|\}, \\ A_2(\xi, \tau) &:= \{(\vec{\xi}, \vec{\tau}) \in \mathbb{Y}_\xi^3 \times \mathbb{R}_\tau^3 : \max A(\xi, \tau, \vec{\xi}, \vec{\tau}) = |\tau_2 + \xi_2^2|\}, \\ A_3(\xi, \tau) &:= \{(\vec{\xi}, \vec{\tau}) \in \mathbb{Y}_\xi^3 \times \mathbb{R}_\tau^3 : \max A(\xi, \tau, \vec{\xi}, \vec{\tau}) = |\tau_3 - \xi_3^2|\} \end{aligned}$$

and

$$\begin{aligned} M(\xi, \tau, \vec{\xi}, \vec{\tau}) &:= \frac{\langle \xi \rangle^{1/2} i \xi_3}{\langle \tau + \xi^2 \rangle^{1/2} \langle \tau_1 + \xi_1^2 \rangle^{1/2} \langle \tau_2 + \xi_2^2 \rangle^{1/2} \langle \tau_3 - \xi_3^2 \rangle^{1/2} \langle \xi_1 \rangle^{1/2} \langle \xi_2 \rangle^{1/2} \langle \xi_3 \rangle^{1/2}}, \\ M_0(\xi, \tau, \vec{\xi}, \vec{\tau}) &:= \frac{\mathbb{1}_{A_0(\xi, \tau)}(\vec{\xi}, \vec{\tau})}{\langle \tau_1 + \xi_1^2 \rangle^{1/2} \langle \tau_2 + \xi_2^2 \rangle^{1/2} \langle \tau_3 - \xi_3^2 \rangle^{1/2} \langle \xi_1 \rangle^{1/2} \langle \xi_2 \rangle^{1/2}}, \\ M_1(\xi, \tau, \vec{\xi}, \vec{\tau}) &:= \frac{\mathbb{1}_{A_1(\xi, \tau)}(\vec{\xi}, \vec{\tau})}{\langle \tau + \xi^2 \rangle^{1/2} \langle \tau_2 + \xi_2^2 \rangle^{1/2} \langle \tau_3 - \xi_3^2 \rangle^{1/2} \langle \xi_1 \rangle^{1/2} \langle \xi_2 \rangle^{1/2}}, \\ M_2(\xi, \tau, \vec{\xi}, \vec{\tau}) &:= \frac{\mathbb{1}_{A_2(\xi, \tau)}(\vec{\xi}, \vec{\tau})}{\langle \tau + \xi^2 \rangle^{1/2} \langle \tau_1 + \xi_1^2 \rangle^{1/2} \langle \tau_3 - \xi_3^2 \rangle^{1/2} \langle \xi_1 \rangle^{1/2} \langle \xi_2 \rangle^{1/2}}, \\ M_3(\xi, \tau, \vec{\xi}, \vec{\tau}) &:= \frac{\mathbb{1}_{A_3(\xi, \tau)}(\vec{\xi}, \vec{\tau})}{\langle \tau + \xi^2 \rangle^{1/2} \langle \tau_1 + \xi_1^2 \rangle^{1/2} \langle \tau_2 + \xi_2^2 \rangle^{1/2} \langle \xi_1 \rangle^{1/2} \langle \xi_2 \rangle^{1/2}}, \\ M_4(\xi, \tau, \vec{\xi}, \vec{\tau}) &:= \frac{1}{\langle \tau + \xi^2 \rangle^{7/16} \langle \tau_1 + \xi_1^2 \rangle^{7/16} \langle \tau_2 + \xi_2^2 \rangle^{7/16} \langle \tau_3 - \xi_3^2 \rangle^{7/16}}, \\ \tilde{M}(\xi, \tau, \vec{\xi}, \vec{\tau}) &:= \frac{M(\xi, \tau, \vec{\xi}, \vec{\tau})}{\langle \tau + \xi^2 \rangle^{1/2}}, \\ \tilde{M}_0(\xi, \tau, \vec{\xi}, \vec{\tau}) &:= \frac{\mathbb{1}_{A_0(\xi, \tau)}(\vec{\xi}, \vec{\tau})}{\langle \tau_1 + \xi_1^2 \rangle^{\frac{1}{2} + \delta} \langle \tau_2 + \xi_2^2 \rangle^{\frac{1}{2} + \delta} \langle \tau_3 - \xi_3^2 \rangle^{\frac{1}{2} + \delta} \langle \xi \rangle^{\frac{1}{2} - 3\delta} \langle \xi_1 \rangle^{\frac{1}{2}} \langle \xi_2 \rangle^{\frac{1}{2}} \langle \xi_3 \rangle^{\frac{1}{2} - 3\delta}}, \\ \tilde{M}_j(\xi, \tau, \vec{\xi}, \vec{\tau}) &:= \frac{M_j(\xi, \tau, \vec{\xi}, \vec{\tau})}{\langle \tau + \xi^2 \rangle^{1/2}}, \quad j \in \{1, 2, 3, 4\}, \end{aligned}$$

where we will choose a  $\delta \in (0, \frac{1}{6})$ .

**Lemma 4.2.** Let  $u_j \in \mathcal{S}(\mathbb{X} \times \mathbb{R})$  such that  $\text{supp } u_j \subseteq \mathbb{X} \times [-T, T]$ ,  $T \in (0, 1]$ ,  $j \in \{1, 2, 3\}$ , and  $f_j(\xi, \tau) := \langle \tau + \xi^2 \rangle^{1/2} \langle \xi \rangle^{1/2} \widehat{u}_j(\xi, \tau)$  for  $j \in \{1, 2\}$  and  $f_3(\xi, \tau) := \langle \tau - \xi^2 \rangle^{1/2} \langle \xi \rangle^{1/2} \widehat{u}_3(\xi, \tau)$ . We have

$$|M| \lesssim \sum_{j=0}^4 M_j, \quad |\tilde{M}| \lesssim \sum_{j=0}^4 \tilde{M}_j \quad (11)$$



and

$$\begin{aligned} & \left\| \int_{\mathbb{R}_\tau^3} \int_{\mathbb{Y}_\xi^3} M_0(\xi, \tau, \vec{\xi}, \vec{\tau}) f_1(\xi_1, \tau_1) f_2(\xi_2, \tau_2) f_3(\xi_3, \tau_3) d\vec{\xi} d\vec{\tau} \right\|_{L_{\xi, \tau}^2} \\ & \lesssim \|u_1\|_{X^{\frac{3}{8}, \frac{3}{8}}} \|u_2\|_{X^{\frac{3}{8}, \frac{3}{8}}} \|u_3\|_{X^{\frac{1}{2}, \frac{1}{2}, -}}, \end{aligned} \quad (12)$$

$$\begin{aligned} & \left\| \int_{\mathbb{R}_\tau^3} \int_{\mathbb{Y}_\xi^3} M_1(\xi, \tau, \vec{\xi}, \vec{\tau}) f_1(\xi_1, \tau_1) f_2(\xi_2, \tau_2) f_3(\xi_3, \tau_3) d\vec{\xi} d\vec{\tau} \right\|_{L_{\xi, \tau}^2} \\ & \lesssim \|u_1\|_{X^{\frac{3}{8}, \frac{1}{2}}} \|u_2\|_{X^{\frac{3}{8}, \frac{3}{8}}} \|u_3\|_{X^{\frac{1}{2}, \frac{1}{2}, -}}, \end{aligned} \quad (13)$$

$$\begin{aligned} & \left\| \int_{\mathbb{R}_\tau^3} \int_{\mathbb{Y}_\xi^3} M_2(\xi, \tau, \vec{\xi}, \vec{\tau}) f_1(\xi_1, \tau_1) f_2(\xi_2, \tau_2) f_3(\xi_3, \tau_3) d\vec{\xi} d\vec{\tau} \right\|_{L_{\xi, \tau}^2} \\ & \lesssim \|u_1\|_{X^{\frac{3}{8}, \frac{3}{8}}} \|u_2\|_{X^{\frac{3}{8}, \frac{1}{2}}} \|u_3\|_{X^{\frac{1}{2}, \frac{1}{2}, -}}, \end{aligned} \quad (14)$$

$$\begin{aligned} & \left\| \int_{\mathbb{R}_\tau^3} \int_{\mathbb{Y}_\xi^3} M_3(\xi, \tau, \vec{\xi}, \vec{\tau}) f_1(\xi_1, \tau_1) f_2(\xi_2, \tau_2) f_3(\xi_3, \tau_3) d\vec{\xi} d\vec{\tau} \right\|_{L_{\xi, \tau}^2} \\ & \lesssim \|u_1\|_{X^{\frac{3}{8}, \frac{3}{8}}} \|u_2\|_{X^{\frac{3}{8}, \frac{3}{8}}} \|u_3\|_{X^{\frac{1}{2}, \frac{1}{2}, -}}, \end{aligned} \quad (15)$$

$$\begin{aligned} & \left\| \int_{\mathbb{R}_\tau^3} \int_{\mathbb{Y}_\xi^3} M_4(\xi, \tau, \vec{\xi}, \vec{\tau}) f_1(\xi_1, \tau_1) f_2(\xi_2, \tau_2) f_3(\xi_3, \tau_3) d\vec{\xi} d\vec{\tau} \right\|_{L_{\xi, \tau}^2} \\ & \lesssim \|u_1\|_{X^{\frac{1}{2}, \frac{15}{32}}} \|u_2\|_{X^{\frac{1}{2}, \frac{15}{32}}} \|u_3\|_{X^{\frac{1}{2}, \frac{15}{32}, -}}. \end{aligned} \quad (16)$$

*Proof.* In the periodic setting, these statements have already been shown in [13, Lemma 4.1, Lemma 4.2, Thm. 4.1]. In that case, for (11), we only had to consider  $\vec{\xi} \in \mathbb{Z}^3$ . But in the non-periodic setting we need to consider  $\vec{\xi} \in \mathbb{R}^3$ . This means, we have to modify the proof of [13] slightly. For the sake of completeness, we will show all these statements simultaneously for the periodic and non-periodic setting.

Let  $(\xi, \tau) \in \mathbb{Y} \times \mathbb{R}$ ,  $\vec{\xi} \in \mathbb{Y}_\xi^3$  and  $\vec{\tau} \in \mathbb{R}_\tau^3$ . Since  $\tau + \xi^2 - (\tau_1 + \xi_1^2 + \tau_2 + \xi_2^2 + \tau_3 - \xi_3^2) = 2(\xi - \xi_1)(\xi - \xi_2)$ , an application of the triangle inequality shows that

$$\begin{aligned} \langle (\xi - \xi_1)(\xi - \xi_2) \rangle^{1/2} & \leq 4 \left( \mathbb{1}_{A_0(\xi, \tau)}(\vec{\xi}, \vec{\tau}) \langle \tau + \xi^2 \rangle^{1/2} + \mathbb{1}_{A_1(\xi, \tau)}(\vec{\xi}, \vec{\tau}) \langle \tau_1 + \xi_1^2 \rangle^{1/2} \right. \\ & \quad + \mathbb{1}_{A_2(\xi, \tau)}(\vec{\xi}, \vec{\tau}) \langle \tau_2 + \xi_2^2 \rangle^{1/2} \\ & \quad \left. + \mathbb{1}_{A_3(\xi, \tau)}(\vec{\xi}, \vec{\tau}) \langle \tau_3 - \xi_3^2 \rangle^{1/2} \right). \end{aligned} \quad (17)$$

We consider the following four cases:

(i)  $|\xi| > 2|\xi_1|$  and  $|\xi| > 2|\xi_2|$ : Here,  $|\xi_3| \lesssim |\xi|$  and  $\langle (\xi - \xi_1)(\xi - \xi_2) \rangle \gtrsim \langle \xi \rangle^2$ . Hence, by (17),

$$|M(\xi, \tau, \vec{\xi}, \vec{\tau})| \lesssim \sum_{j=0}^3 M_j(\xi, \tau, \vec{\xi}, \vec{\tau}).$$

(ii)  $|\xi| \leq 2|\xi_1|$  and  $|\xi| \leq 2|\xi_2|$ : In this case, we have  $|\xi_3| \lesssim \max\{|\xi_1|, |\xi_2|\}$  and  $|\xi| \leq 2 \min\{|\xi_1|, |\xi_2|\}$ . This means

$$|M(\xi, \tau, \vec{\xi}, \vec{\tau})| \lesssim M_4(\xi, \tau, \vec{\xi}, \vec{\tau}).$$

(iii)  $|\xi| > 2|\xi_1|$  and  $|\xi| \leq 2|\xi_2|$ : Since  $|\xi| \leq |\xi - \xi_1| + \frac{1}{2}|\xi|$ , we have  $|\xi| \leq 2|\xi - \xi_1|$  and therefore  $2\langle (\xi - \xi_1)(\xi - \xi_2) \rangle \geq |\xi| \cdot |\xi - \xi_2|$ . Furthermore,  $\langle \xi \rangle^{1/2} \leq 1 + |\xi|^{1/2}$  and

$|\xi_3|^{1/2} \leq |\xi - \xi_2|^{1/2} + |\xi_1|^{1/2}$ . Using  $\langle \xi_1 \rangle \geq 1$  and (17) provides

$$|M(\xi, \tau, \vec{\xi}, \vec{\tau})| \lesssim \sum_{j=0}^4 M_j(\xi, \tau, \vec{\xi}, \vec{\tau}).$$

(iv)  $|\xi| \leq 2|\xi_1|$  and  $|\xi| > 2|\xi_2|$ : By symmetry, this is a direct consequence of (iii).

From the first estimate of (11), we obtain

$$|\tilde{M}(\xi, \tau, \vec{\xi}, \vec{\tau})| \lesssim \langle \tau + \xi^2 \rangle^{-1/2} M_0(\xi, \tau, \vec{\xi}, \vec{\tau}) + \sum_{j=1}^4 \tilde{M}_j(\xi, \tau, \vec{\xi}, \vec{\tau}).$$

This means, the second estimate of (11) follows from  $\langle \tau + \xi^2 \rangle^{-1/2} M_0(\xi, \tau, \vec{\xi}, \vec{\tau}) \lesssim \tilde{M}_0(\xi, \tau, \vec{\xi}, \vec{\tau})$ . Therefore, we consider again the four cases from above:

(i)  $|\xi| > 2\xi_1$  and  $|\xi| > 2|\xi_2|$ : Here,  $|\xi_3| \lesssim |\xi|$ . For  $(\vec{\xi}, \vec{\tau}) \in A_0(\xi, \tau)$ , resonance relation (10) implies  $|\tau + \xi^2| \gtrsim |\xi| \cdot |\xi_3|$ . Hence

$$\langle \tau + \xi^2 \rangle^{1/2} \gtrsim \langle \tau_1 + \xi_1^2 \rangle^\delta \langle \tau_2 + \xi_2^2 \rangle^\delta \langle \tau_3 - \xi_3^2 \rangle^\delta \langle \xi \rangle^{\frac{1}{2}-3\delta} \langle \xi_3 \rangle^{\frac{1}{2}-3\delta}$$

and consequently  $\langle \tau + \xi^2 \rangle^{-1/2} M_0(\xi, \tau, \vec{\xi}, \vec{\tau}) \lesssim \tilde{M}_0(\xi, \tau, \vec{\xi}, \vec{\tau})$ .

(ii)  $|\xi| \leq 2|\xi_1|$  and  $|\xi| \leq 2|\xi_2|$ : In this case, we have  $|\xi_3| \lesssim \max\{|\xi_1|, |\xi_2|\}$  and  $|\xi| \lesssim 2 \min\{|\xi_1|, |\xi_2|\}$  which implies  $|\tilde{M}| \lesssim \tilde{M}_4$ .

(iii)  $|\xi| > 2|\xi_1|$  and  $|\xi| \leq 2|\xi_2|$ : First, let  $|\xi| < 1$ . Then  $|\xi_1| < \frac{1}{2}$  and  $|\xi_2 + \xi_3| < \frac{3}{2}$ . This means  $|\xi_3| < \frac{3}{2} + |\xi_2|$  and  $\langle \xi_3 \rangle \lesssim \langle \xi_2 \rangle$ . Since  $|\xi| < 1$ ,  $|\xi_1| < \frac{1}{2}$ , we obtain

$$|\tilde{M}(\xi, \tau, \vec{\xi}, \vec{\tau})| \lesssim \tilde{M}_4(\xi, \tau, \vec{\xi}, \vec{\tau}).$$

Secondly, let  $|\xi_3| < 1$ . Then  $\langle \xi_3 \rangle \sim 1$  and from  $|\xi| \leq 2|\xi_2|$ ,  $\langle \xi_1 \rangle \geq 1$ , we obtain

$$|\tilde{M}(\xi, \tau, \vec{\xi}, \vec{\tau})| \lesssim \tilde{M}_4(\xi, \tau, \vec{\xi}, \vec{\tau}).$$

Thirdly, assume that  $|\xi_1| > |\xi - \xi_2|$ . Then  $|\xi_3| \leq |\xi - \xi_2| + |\xi_1| < 2|\xi_1|$  and  $|\xi| \leq 2|\xi_2|$  which implies

$$|\tilde{M}(\xi, \tau, \vec{\xi}, \vec{\tau})| \lesssim \tilde{M}_4(\xi, \tau, \vec{\xi}, \vec{\tau}).$$

Finally, we have to consider the case  $|\xi|, |\xi_3| \geq 1$  and  $|\xi_1| \leq |\xi - \xi_2|$ . Here,  $\langle \xi \rangle \sim |\xi|$ ,  $\langle \xi_3 \rangle \sim |\xi_3|$  and the triangle inequality provides

$$|\xi_3| \leq |\xi - \xi_2| + |\xi_1| \leq 2|\xi - \xi_2|.$$

Furthermore, we have

$$|\xi| \leq 2|\xi - \xi_1|, \quad |\xi| \cdot |\xi - \xi_2| \leq 2\langle (\xi - \xi_1)(\xi - \xi_2) \rangle$$

and as a consequence  $\langle \xi \rangle \langle \xi_3 \rangle \lesssim \langle (\xi - \xi_1)(\xi - \xi_2) \rangle$ . For  $(\vec{\xi}, \vec{\tau}) \in A_0(\xi, \tau)$ , resonance relation (10) implies

$$\begin{aligned} \langle \tau + \xi^2 \rangle^{1/2} &\gtrsim \langle (\xi - \xi_1)(\xi - \xi_2) \rangle^{\frac{1}{2}-3\delta} \langle \tau_1 + \xi_1^2 \rangle^\delta \langle \tau_2 + \xi_2^2 \rangle^\delta \langle \tau_3 - \xi_3^2 \rangle^\delta \\ &\gtrsim \langle \xi \rangle^{\frac{1}{2}-3\delta} \langle \xi_3 \rangle^{\frac{1}{2}-3\delta} \langle \tau_1 + \xi_1^2 \rangle^\delta \langle \tau_2 + \xi_2^2 \rangle^\delta \langle \tau_3 - \xi_3^2 \rangle^\delta, \end{aligned}$$

so that  $\langle \tau + \xi^2 \rangle^{-1/2} M_0(\xi, \tau, \vec{\xi}, \vec{\tau}) \lesssim \tilde{M}_0(\xi, \tau, \vec{\xi}, \vec{\tau})$ .

(iv)  $|\xi| \leq 2|\xi_1|$  and  $|\xi| > 2|\xi_2|$ : This is again a consequence of case (iii).

Now, we prove (12)-(16): By definition of  $M_0$ , Hölder's inequality and estimates (4), (6), we obtain

$$\begin{aligned} \left\| \int_{\mathbb{R}_\tau^3} \int_{\mathbb{Y}_\xi^3} M_0(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_j(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L_{\xi, \tau}^2} &\leq \|u_1 \cdot u_2 \cdot J_x^{1/2} u_3\|_{L_{t,x}^2} \\ &\lesssim \|u_1\|_{X^{\frac{3}{8}, \frac{3}{8}}} \|u_2\|_{X^{\frac{3}{8}, \frac{3}{8}}} \|u_3\|_{X^{\frac{1}{2}, \frac{1}{2}, -}}. \end{aligned}$$

For  $M_1$ , we denote

$$\left\| \int_{\mathbb{R}_\tau^3} \int_{\mathbb{Y}_\xi^3} M_1(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_j(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L_{\xi, \tau}^2} \leq \|\Gamma^{1/2} u_1 \cdot u_2 \cdot J_x^{1/2} u_3\|_{X^{0, -\frac{1}{2}}}.$$

Applying (5), Hölder's inequality, (4) and (6) provides

$$\begin{aligned} \|\Gamma^{1/2} u_1 \cdot u_2 \cdot J_x^{1/2} u_3\|_{X^{0, -\frac{3}{8}}} &\lesssim \|\Gamma^{1/2} u_1 \cdot u_2 \cdot J_x^{1/2} u_3\|_{L_t^{8/7} L_x^2} \\ &\leq \|\Gamma^{1/2} u_1\|_{L_t^2 L_x^8} \|u_2\|_{L_{t,x}^8} \|J_x^{1/2} u_3\|_{L_{t,x}^4} \\ &\lesssim \|u_1\|_{X^{\frac{3}{8}, \frac{1}{2}}} \|u_2\|_{X^{\frac{3}{8}, \frac{3}{8}}} \|u_3\|_{X^{\frac{1}{2}, \frac{1}{2}, -}}. \end{aligned} \quad (18)$$

Since  $\|\cdot\|_{X^{0, -\frac{1}{2}}} \leq \|\cdot\|_{X^{0, -\frac{3}{8}}}$ , we obtain the estimate for  $M_1$ .

Because of symmetry, the estimate for  $M_2$  can be shown analogously.

By definition of  $M_3$ ,

$$\left\| \int_{\mathbb{R}_\tau^3} \int_{\mathbb{Y}_\xi^3} M_3(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_j(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L_{\xi, \tau}^2} \leq \|u_1 \cdot u_2 \cdot J_x^{1/2} \Gamma_-^{1/2} u_3\|_{X^{0, -\frac{1}{2}}}.$$

Using (7), Hölder's inequality and (4) provides

$$\begin{aligned} \|u_1 \cdot u_2 \cdot J_x^{1/2} \Gamma_-^{1/2} u_3\|_{X^{0, -\frac{7}{16}}} &\lesssim \|u_1 \cdot u_2 \cdot J_x^{1/2} \Gamma_-^{1/2} u_3\|_{L_{t,x}^{4/3}} \\ &\lesssim \|u_1\|_{X^{\frac{3}{8}, \frac{3}{8}}} \|u_2\|_{X^{\frac{3}{8}, \frac{3}{8}}} \|u_3\|_{X^{\frac{1}{2}, \frac{1}{2}, -}}. \end{aligned} \quad (19)$$

Since  $\|\cdot\|_{X^{0, -\frac{1}{2}}} \leq \|\cdot\|_{X^{0, -\frac{7}{16}}}$ , we obtain the conclusion for  $M_3$ .

For  $M_4$ , we have

$$\begin{aligned} \left\| \int_{\mathbb{R}_\tau^3} \int_{\mathbb{Y}_\xi^3} M_4(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_j(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L_{\xi, \tau}^2} \\ \leq \|J_x^{1/2} \Gamma^{1/16} u_1 \cdot J_x^{1/2} \Gamma^{1/16} u_2 \cdot J_x^{1/2} \Gamma_-^{1/16} u_3\|_{X^{0, -\frac{7}{16}}}. \end{aligned}$$

The dual Strichartz estimate (7), Hölder's inequality and Strichartz estimate (6) imply

$$\begin{aligned} &\|J_x^{1/2} \Gamma^{1/16} u_1 \cdot J_x^{1/2} \Gamma^{1/16} u_2 \cdot J_x^{1/2} \Gamma_-^{1/16} u_3\|_{X^{0, -\frac{7}{16}}} \\ &\lesssim \|J_x^{1/2} \Gamma^{1/16} u_1 \cdot J_x^{1/2} \Gamma^{1/16} u_2 \cdot J_x^{1/2} \Gamma_-^{1/16} u_3\|_{L_{t,x}^{4/3}} \\ &\lesssim \|u_1\|_{X^{\frac{1}{2}, \frac{15}{32}}} \|u_2\|_{X^{\frac{1}{2}, \frac{15}{32}}} \|u_3\|_{X^{\frac{1}{2}, \frac{15}{32}, -}} \end{aligned} \quad (20)$$

which proves the estimate for  $M_4$ .  $\square$

**Remark.** Similar estimates for  $\tilde{M}_j$  will be shown in the proof of theorem 4.6.

To prove the trilinear estimate, we follow the ideas of [13, Thm. 4.1 and Thm. 4.2] for the  $H^s$ -case. In order to localize frequencies, we write  $u_j = \sum_{N_j \in \mathcal{D}_1} P_{N_j} u_j$ . We need to eliminate the sums over  $N_j \in \mathcal{D}_1$ . Therefore, the following elementary estimates are useful:

**Lemma 4.4.** *Let  $u \in \mathcal{S}(\mathbb{X} \times \mathbb{R})$ ,  $\delta > 0$  and  $s, b \in \mathbb{R}$ . Then*

$$\sum_{N \in \mathcal{D}_1} N^{-\delta} \|P_N u\|_{X^{s,b,\pm}} \lesssim_\delta \|u\|_{\mathfrak{X}^{s,b,\pm}}, \quad (21)$$

$$\sum_{N \in \mathcal{D}_1} \|P_N u\|_{X^{s,b}} \lesssim_\delta \|u\|_{\mathfrak{X}^{s+\delta,b}}. \quad (22)$$

For  $N \in \mathcal{D}_1$ , we have

$$\sum_{\mathcal{D}_1 \ni N_1 \sim N} \|P_{N_1} u\|_{X^{s,b,\pm}} \lesssim \|u\|_{\mathfrak{X}^{s,b,\pm}} \quad (23)$$

with an implicit constant which does not depend on  $N$ .

For  $k \geq 1$ , we have

$$\sum_{\mathcal{D}_1 \ni N \leq k} \|P_N u\|_{X^{s,b,\pm}} \lesssim_k \|u\|_{\mathfrak{X}^{s,b,\pm}}. \quad (24)$$

*Proof.* Estimate (21) is a direct consequence of the convergence of the geometric series  $\sum_{N \in \mathcal{D}_1} N^{-\delta}$ . Estimate (22) follows from (21).

Let  $C > 1$  the implicit constant corresponding to  $\sim$ . There are  $2\lfloor \log_2 C \rfloor + 1$  dyadic  $N_1$  with  $N_1 \sim N$  which implies (23).

Finally, for  $k \geq 1$ , there are  $\lfloor \log_2 k \rfloor$  dyadic  $N$  such that  $1 < N \leq k$  and we obtain (24).  $\square$

**Remark.** One can show the same estimates for  $Y^{s,b}$ - and  $\mathcal{Y}^{s,b}$ -norms. But because of (3), we need these statements only for  $X^{s,b}$  and  $\mathfrak{X}^{s,b}$ .

**Theorem 4.5: Trilinear estimate for  $\mathfrak{X}^{\frac{1}{2},-\frac{1}{2}}$ .** *Let  $T \in (0, 1]$ ,  $u_j \in \mathcal{S}(\mathbb{X} \times \mathbb{R})$  with  $\text{supp } u_j \subseteq \mathbb{X} \times (-T, T)$ ,  $j \in \{1, 2, 3\}$ . There exists an  $\varepsilon > 0$  such that*

$$\|\mathcal{T}(u_1, u_2, u_3)\|_{\mathfrak{X}^{\frac{1}{2},-\frac{1}{2}}} \lesssim T^\varepsilon \|u_1\|_{\mathfrak{X}^{\frac{1}{2},\frac{1}{2}}} \|u_2\|_{\mathfrak{X}^{\frac{1}{2},\frac{1}{2}}} \|u_3\|_{\mathfrak{X}^{\frac{1}{2},\frac{1}{2},-}}. \quad (25)$$

*Proof.* By applying the triangle inequality, we may assume  $P_{N_j} u_j \geq 0$  and  $P_{N_j} u_j = \chi_T(t) P_{N_j} u_j$ . Take  $\varepsilon \in (0, \frac{1}{32})$ . We need to show the estimates

$$\begin{aligned} \|P_1 \mathcal{T}(u_1, u_2, u_3)\|_{X^{\frac{1}{2},-\frac{1}{2}}} &\lesssim T^\varepsilon \|u_1\|_{\mathfrak{X}^{\frac{1}{2},\frac{1}{2}}} \|u_2\|_{\mathfrak{X}^{\frac{1}{2},\frac{1}{2}}} \|u_3\|_{\mathfrak{X}^{\frac{1}{2},\frac{1}{2},-}}, \\ \sup_{N > 1} \|P_N \mathcal{T}(u_1, u_2, u_3)\|_{X^{\frac{1}{2},-\frac{1}{2}}} &\lesssim T^\varepsilon \|u_1\|_{\mathfrak{X}^{\frac{1}{2},\frac{1}{2}}} \|u_2\|_{\mathfrak{X}^{\frac{1}{2},\frac{1}{2}}} \|u_3\|_{\mathfrak{X}^{\frac{1}{2},\frac{1}{2},-}}. \end{aligned}$$

Let

$$\begin{aligned} f_{N_j, u_j}(\xi, \tau) &:= \langle \tau + \xi^2 \rangle^{1/2} \langle \xi \rangle^{1/2} \widehat{P_{N_j} u_j}(\xi, \tau), \quad j \in \{1, 2\}, \\ f_{N_3, u_3}(\xi, \tau) &:= \langle \tau - \xi^2 \rangle^{1/2} \langle \xi \rangle^{1/2} \widehat{P_{N_3} u_3}(\xi, \tau). \end{aligned}$$

We define

$$\Omega_\xi := \begin{cases} \Omega'_\xi := \mathbb{R}_\xi^3 & \text{if } \mathbb{X} = \mathbb{R}, \\ \Omega'_\xi \cup \Omega''_\xi & \text{if } \mathbb{X} = \mathbb{T}, \end{cases}$$

where

$$\Omega'_\xi := \{\vec{\xi} \in \mathbb{Z}_\xi^3 : \xi_1, \xi_2 \neq \xi\}, \quad \Omega''_\xi := \{\vec{\xi} \in \mathbb{Z}_\xi^3 : \xi_1 = \xi_2 = \xi, \xi_3 = -\xi\}$$

for  $\mathbb{X} = \mathbb{T}$  with integration with respect to the counting measure. Then

$$\begin{aligned} & \|P_1 \mathcal{T}(u_1, u_2, u_3)\|_{X^{\frac{1}{2}, -\frac{1}{2}}} \\ &= \left\| \chi_{\leq 1}(\xi) \sum_{N_1, N_2, N_3 \in \mathcal{D}_1} \int_{\mathbb{R}_\tau^3} \int_{\Omega_\xi} M(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_{N_j, u_j}(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L_{\xi, \tau}^2} \end{aligned}$$

and the same for  $P_N \mathcal{T}(u_1, u_2, u_3)$  with dyadic  $N > 1$  by replacing  $\chi_{\leq 1}$  with  $\chi_N$ .

On  $\Omega''_\xi$ , we will only get a positive term if  $N_1, N_2, N_3 \lesssim 1$  and  $N_1, N_2, N_3 \sim N$  respectively. For  $N > 1$ , we use estimates (11), (12)-(16) and (23) to conclude

$$\begin{aligned} & \left\| \chi_N(\xi) \sum_{N_1, N_2, N_3 \in \mathcal{D}_1} \int_{\mathbb{R}_\tau^3} \int_{\Omega''_\xi} M(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_{N_j, u_j}(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L_{\xi, \tau}^2} \\ & \lesssim T^\varepsilon \sum_{N_1, N_2, N_3 \sim N} \|P_{N_1} u_1\|_{X^{\frac{1}{2}, \frac{1}{2}}} \|P_{N_2} u_2\|_{X^{\frac{1}{2}, \frac{1}{2}}} \|P_{N_3} u_3\|_{X^{\frac{1}{2}, \frac{1}{2}, -}} \\ & \lesssim T^\varepsilon \|u_1\|_{X^{\frac{1}{2}, \frac{1}{2}}} \|u_2\|_{X^{\frac{1}{2}, \frac{1}{2}}} \|u_3\|_{X^{\frac{1}{2}, \frac{1}{2}, -}} \end{aligned}$$

with a constant which does not depend on  $N$ . For the small frequencies, we obtain from estimates (11), (12)-(16) and (24) that

$$\begin{aligned} & \left\| \chi_{\leq 1}(\xi) \sum_{N_1, N_2, N_3 \in \mathcal{D}_1} \int_{\mathbb{R}_\tau^3} \int_{\Omega''_\xi} M(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_{N_j, u_j}(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L_{\xi, \tau}^2} \\ & \lesssim T^\varepsilon \sum_{N_1, N_2, N_3 \lesssim 1} \|P_{N_1} u_1\|_{X^{\frac{1}{2}, \frac{1}{2}}} \|P_{N_2} u_2\|_{X^{\frac{1}{2}, \frac{1}{2}}} \|P_{N_3} u_3\|_{X^{\frac{1}{2}, \frac{1}{2}, -}} \\ & \lesssim T^\varepsilon \|u_1\|_{X^{\frac{1}{2}, \frac{1}{2}}} \|u_2\|_{X^{\frac{1}{2}, \frac{1}{2}}} \|u_3\|_{X^{\frac{1}{2}, \frac{1}{2}, -}}. \end{aligned}$$

We still need to consider the set  $\Omega'_\xi$ . The conclusion follows from

$$\begin{aligned} & \left\| \chi_{\leq 1}(\xi) \sum_{N_1, N_2, N_3} \int_{\mathbb{R}_\tau^3} \int_{\Omega'_\xi} M(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_{N_j, u_j}(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L_{\xi, \tau}^2} \\ & \lesssim T^\varepsilon \|u_1\|_{X^{\frac{1}{2}, \frac{1}{2}}} \|u_2\|_{X^{\frac{1}{2}, \frac{1}{2}}} \|u_3\|_{X^{\frac{1}{2}, \frac{1}{2}, -}}, \end{aligned} \quad (26)$$

$$\begin{aligned} & \sup_{N > 1} \left\| \chi_N(\xi) \sum_{N_1, N_2, N_3} \int_{\mathbb{R}_\tau^3} \int_{\Omega'_\xi} M(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_{N_j, u_j}(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L_{\xi, \tau}^2} \\ & \lesssim T^\varepsilon \|u_1\|_{X^{\frac{1}{2}, \frac{1}{2}}} \|u_2\|_{X^{\frac{1}{2}, \frac{1}{2}}} \|u_3\|_{X^{\frac{1}{2}, \frac{1}{2}, -}}. \end{aligned} \quad (27)$$

Estimate (26) can be shown by similar arguments as (27) just by replacing  $\chi_N$  with  $\chi_{\leq 1}$  and " $\sim N$ " with " $\lesssim 1$ ". Hence we will only prove (27).

Let  $N > 1$  be dyadic. By symmetry, we may assume that  $N_1 \leq N_2$ . We distinguish

between the cases  $N_3 \gg N_2$ ,  $N_3 \sim N_2$  and  $N_3 \ll N_2$  (taking an implicit constant greater than 8). We write

$$\left\| \sum_{N_1, N_2, N_3} \chi_N(\xi) \int_{\mathbb{R}_\tau^3} \int_{\Omega'_\xi} M(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_{N_j, u_j}(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L_{\xi, \tau}^2} \lesssim I + II + III$$

taking the sum over  $\|\chi_N(\xi) \int_{\mathbb{R}_\tau^3} \int_{\Omega'_\xi} M(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_{N_j, u_j}(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau}\|_{L_{\xi, \tau}^2}$  with the restrictions

$$I : N_1 \leq N_2 \ll N_3, \quad II : N_1 \leq N_2 \sim N_3, \quad III : N_1 \leq N_2, N_2 \gg N_3.$$

*Case I:*  $N_1 \leq N_2 \ll N_3$ . In order to obtain a positive contribution, we may assume that  $N_3 \sim N$ . Then resonance relation (10) provides

$$\max\{|\tau + \xi^2|, |\tau_1 + \xi_1^2|, |\tau_2 + \xi_2^2|, |\tau_3 - \xi_3^2|\} \gtrsim NN_3.$$

Since  $N, N_3 \geq 1$ , we have  $NN_3 \sim \langle \xi \rangle \langle \xi_3 \rangle$  which implies

$$\begin{aligned} |M(\xi, \tau, \vec{\xi}, \vec{\tau})| &\leq \frac{\langle \xi \rangle^{1/2} \langle \xi_3 \rangle^{1/2}}{\langle \tau + \xi^2 \rangle^{1/2} \langle \tau_1 + \xi_1^2 \rangle^{1/2} \langle \tau_2 + \xi_2^2 \rangle^{1/2} \langle \tau_3 - \xi_3^2 \rangle^{1/2} \langle \xi_1 \rangle^{1/2} \langle \xi_2 \rangle^{1/2}} \\ &\lesssim \sum_{j=0}^3 M_j(\xi, \tau, \vec{\xi}, \vec{\tau}). \end{aligned}$$

By (12)-(15), we obtain

$$\begin{aligned} I &\lesssim T^\varepsilon \sum_{N_1, N_2} \sum_{N_3 \sim N} \|P_{N_1} u_1\|_{X^{\frac{3}{8}, \frac{1}{2}}} \|P_{N_2} u_2\|_{X^{\frac{3}{8}, \frac{1}{2}}} \|P_{N_3} u_3\|_{X^{\frac{1}{2}, \frac{1}{2}, -}} \\ &\lesssim T^\varepsilon \|u_1\|_{\mathfrak{X}^{\frac{1}{2}, \frac{1}{2}}} \|u_2\|_{\mathfrak{X}^{\frac{1}{2}, \frac{1}{2}}} \|u_3\|_{\mathfrak{X}^{\frac{1}{2}, \frac{1}{2}, -}}. \end{aligned}$$

*Case II:*  $N_1 \leq N_2 \sim N_3$ . We distinguish between

$$IIa : N_1 \sim N_2, \quad IIb : N_1 \ll N_2.$$

*Case IIa:*  $N_1 \sim N_2 \sim N_3$ . For a positive contribution, we need  $N_1, N_2, N_3 \sim N$ . Therefore, we get the desired estimate directly by (12)-(16).

*Case IIb:*  $N_1 \ll N_2 \sim N_3$ . We consider the cases

$$IIb_1 : N \lesssim N_1, \quad IIb_2 : N \gg N_1.$$

*Case IIb<sub>1</sub>:*  $N \lesssim N_1 \ll N_2 \sim N_3$ . Here

$$|M(\xi, \tau, \vec{\xi}, \vec{\tau})| \lesssim \frac{1}{\langle \tau + \xi^2 \rangle^{1/2} \langle \tau_1 + \xi_1^2 \rangle^{1/2} \langle \tau_2 + \xi_2^2 \rangle^{1/2} \langle \tau_3 - \xi_3^2 \rangle^{1/2}}. \quad (28)$$

We subdivide  $\Omega'_\xi = \Omega'_{\xi, +} \cup \Omega'_{\xi, -}$ , where  $\Omega'_{\xi, +} := \{\vec{\xi} \in \Omega'_\xi : |\xi - \xi_1| \geq N_3^{-1/2}\}$  and  $\Omega'_{\xi, -} := \Omega'_\xi \setminus \Omega'_{\xi, +}$ . Then we can split  $IIb_1$  into  $IIb_{1,+}$  and  $IIb_{1,-}$  with

$$IIb_{1,\pm} := \sum_{\substack{N_1, N_2, N_3 \\ N \lesssim N_1 \ll N_2 \sim N_3}} \left\| \chi_N(\xi) \int_{\mathbb{R}_\tau^3} \int_{\Omega'_{\xi, \pm}} M(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_{N_j, u_j}(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L_{\xi, \tau}^2}.$$

*Case IIb<sub>1,+</sub>:*  $N \lesssim N_1 \ll N_2 \sim N_3$  and  $|\xi - \xi_1| \geq N_3^{-1/2}$ . Due to resonance relation (10), we may assume that

$$\max\{|\tau + \xi^2|, |\tau_1 + \xi_1^2|, |\tau_2 + \xi_2^2|, |\tau_3 - \xi_3^2|\} \gtrsim N_3^{1/2} \gtrsim N_1^{1/4} N_3^{1/4}.$$

By (28), we obtain

$$|M(\xi, \tau, \vec{\xi}, \vec{\tau})| \lesssim N_1^{-1/64} N_3^{-1/64} M_4(\xi, \tau, \vec{\xi}, \vec{\tau}) \quad (29)$$

and the conclusion follows from (16).

*Case IIb<sub>1,-</sub>*:  $N \lesssim N_1 \ll N_2 \sim N_3$  and  $|\xi - \xi_1| < N_3^{-1/2}$ . In the periodic setting, we have  $\xi, \xi_1 \in \mathbb{Z}$  and  $N_3^{-1/2} \leq 1$  implies  $\xi = \xi_1$ . Hence  $\vec{\xi} \notin \Omega'_\xi$  and  $\Omega'_{\xi,-} = \emptyset$ .

For the non-periodic setting, we introduce

$$\begin{aligned} \rho_{N_3, \xi} &:= \mathbb{1}_{(\xi - N_3^{-1/2}, \xi + N_3^{-1/2})}, \\ I_k &:= (kN_3^{-1/2}, (k+1)N_3^{-1/2}). \end{aligned}$$

Estimate (28) implies

$$\begin{aligned} & \left\| \chi_N(\xi) \int_{\mathbb{R}_\tau^3} \int_{\Omega'_{\xi,-}} M(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_{N_j, u_j}(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L^2_{\xi, \tau}} \\ & \lesssim \left\| \langle \tau + \xi^2 \rangle^{-1/2} \int_{\mathbb{R}_{\xi, \tau}^6} \rho_{N_3, \xi}(\xi_1) \prod_{j=1}^3 \langle \xi_j \rangle^{1/2} \widehat{P_{N_j} u_j}(\xi_j, \tau_j) d(\vec{\xi}, \vec{\tau}) \right\|_{L^2_{\xi, \tau}} \\ & \leq \sum_{l \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \left\| \langle \tau + \xi^2 \rangle^{-1/2} \int_{\mathbb{R}_{\xi, \tau}^6} \mathbb{1}_{I_k}(\xi_2) \mathbb{1}_{I_l}(\xi_3) \rho_{N_3, \xi}(\xi_1) \right. \\ & \quad \cdot \left. \prod_{j=1}^3 \langle \xi_j \rangle^{1/2} \widehat{P_{N_j} u_j}(\xi_j, \tau_j) d(\vec{\xi}, \vec{\tau}) \right\|_{L^2_{\xi, \tau}}. \quad (30) \end{aligned}$$

For a positive contribution, we need  $\xi_2 \in I_k$  and  $|\xi_2 + \xi_3| = |\xi - \xi_1| \leq N_3^{-1/2}$ . Hence

$$\xi_3 \in [-\xi_2 - N_3^{-1/2}, -\xi_2 + N_3^{-1/2}] \subseteq [-(k+2)N_3^{-1/2}, -(k-1)N_3^{-1/2}]$$

and dividing the integration region into  $\langle \tau_2 + \xi_2^2 \rangle \leq \langle \tau_3 - \xi_3^2 \rangle$ ,  $\langle \tau_2 + \xi_2^2 \rangle > \langle \tau_3 - \xi_3^2 \rangle$  provides

$$\begin{aligned} (30) & \lesssim \sum_{l=0}^2 \sum_{k \in \mathbb{Z}} \left\| \langle \tau + \xi^2 \rangle^{-1/2} \int_{\mathbb{R}_{\xi, \tau}^6} \mathbb{1}_{I_k}(\xi_2) \mathbb{1}_{I_{-(k+l)}}(\xi_3) \right. \\ & \quad \cdot \prod_{j=1}^3 \langle \xi_j \rangle^{1/2} \widehat{P_{N_j} u_j}(\xi_j, \tau_j) \cdot \langle \tau_2 + \xi_2^2 \rangle^{-1/16} \langle \tau_3 - \xi_3^2 \rangle^{1/16} d(\vec{\xi}, \vec{\tau}) \left. \right\|_{L^2_{\xi, \tau}} \\ & \quad + \sum_{l=0}^2 \sum_{k \in \mathbb{Z}} \left\| \langle \tau + \xi^2 \rangle^{-1/2} \int_{\mathbb{R}_{\xi, \tau}^6} \mathbb{1}_{I_k}(\xi_2) \mathbb{1}_{I_{-(k+l)}}(\xi_3) \right. \\ & \quad \cdot \prod_{j=1}^3 \langle \xi_j \rangle^{1/2} \widehat{P_{N_j} u_j}(\xi_j, \tau_j) \cdot \langle \tau_2 + \xi_2^2 \rangle^{1/16} \langle \tau_3 - \xi_3^2 \rangle^{-1/16} d(\vec{\xi}, \vec{\tau}) \left. \right\|_{L^2_{\xi, \tau}} \\ & =: A(N_1, N_2, N_3) + B(N_1, N_2, N_3). \end{aligned}$$

An application of estimate (7) and the inequalities of Hölder and Cauchy-Schwarz leads to

$$A(N_1, N_2, N_3) \lesssim \left\| J_x^{1/2} P_{N_1} u_1 \right\|_{L_{t,x}^4} \sum_{l=0}^2 \left( \sum_{k \in \mathbb{Z}} \left\| P_{I_k} (J_x^{1/2} \Gamma^{-1/16} P_{N_2} u_2) \right\|_{L_{t,x}^4}^2 \right)^{1/2} \cdot \left( \sum_{k \in \mathbb{Z}} \left\| P_{I_{-(k+l)}} (J_x^{1/2} \Gamma^{-1/16} P_{N_3} u_3) \right\|_{L_{t,x}^4}^2 \right)^{1/2}. \quad (31)$$

Bernstein's inequality,  $|I_k| = N_3^{-1/2}$ ,  $\text{supp } u \subseteq \mathbb{R} \times [-T, T]$  and Strichartz estimate (8) provide

$$\begin{aligned} \left\| P_{I_k} (J_x^{1/2} \Gamma^{-1/16} P_{N_2} u_2) \right\|_{L_{t,x}^4} &\lesssim N_3^{-\frac{1}{2}(\frac{1}{2}-\frac{1}{4})} \left\| P_{I_k} (J_x^{1/2} P_{N_2} u_2) \right\|_{L_t^4 L_x^2} \\ &\lesssim T^{1/4} N_3^{-\frac{1}{8}} \left\| P_{I_k} (J_x^{1/2} \Gamma^{-1/16} P_{N_2} u_2) \right\|_{L_t^\infty L_x^2} \\ &\lesssim T^{1/4} N_3^{-\frac{1}{8}} \left\| P_{I_k} (J_x^{1/2} \Gamma^{-1/16} P_{N_2} u_2) \right\|_{X^{0, \frac{9}{16}}}. \end{aligned} \quad (32)$$

Applying estimate (32) to the second factor of (31) and (6) to the other factors yields

$$\begin{aligned} A(N_1, N_2, N_3) &\lesssim T^{1/4} N_3^{-1/8} \|P_{N_1} u_1\|_{X^{\frac{1}{2}, \frac{1}{2}}} \sum_{l=0}^2 \left( \sum_{k \in \mathbb{Z}} \|P_{I_k} P_{N_2} u_2\|_{X^{\frac{1}{2}, \frac{1}{2}}}^2 \right)^{1/2} \\ &\quad \cdot \left( \sum_{k \in \mathbb{Z}} \|P_{I_{-(k+l)}} P_{N_3} u_3\|_{X^{\frac{1}{2}, \frac{1}{2}, -}}^2 \right)^{1/2} \\ &\lesssim T^{1/4} N_1^{-1/16} N_3^{-1/16} \|P_{N_1} u_1\|_{X^{\frac{1}{2}, \frac{1}{2}}} \|P_{N_2} u_2\|_{X^{\frac{1}{2}, \frac{1}{2}}} \|P_{N_3} u_3\|_{X^{\frac{1}{2}, \frac{1}{2}, -}}, \end{aligned}$$

where we used almost orthogonality and  $N_1 \lesssim N_3$ .

For  $B(N_1, N_2, N_3)$ , we obtain the same upper bound by switching between  $u_2$  and  $u_3$  and applying Bernstein's inequality on  $\|P_{I_{-(k+l)}} P_{N_3} u_3\|_{L_{t,x}^4}$ .

Overall, we have shown that

$$\begin{aligned} IIb_{1,-} &\lesssim \sum_{N_1, N_3} \sum_{N_2 \sim N_3} (A(N_1, N_2, N_3) + B(N_1, N_2, N_3)) \\ &\lesssim T^{1/4} \|u_1\|_{X^{\frac{1}{2}, \frac{1}{2}}} \|u_2\|_{X^{\frac{1}{2}, \frac{1}{2}}} \|u_3\|_{X^{\frac{1}{2}, \frac{1}{2}, -}}. \end{aligned}$$

*Case IIb<sub>2</sub>:*  $N_1 \ll N$  and  $N_1 \ll N_2 \sim N_3$ . We have  $|\xi_1 + \xi_3| \sim N_3$ ,  $|\xi_2 + \xi_3| = |\xi - \xi_1| \sim N$  and by resonance relation (10), we can suppose that

$$\max\{|\tau + \xi^2|, |\tau_1 + \xi_1^2|, |\tau_2 + \xi_2^2|, |\tau_3 - \xi_3^2|\} \gtrsim N_3 N.$$

Since  $N_3 \sim N_2$ , the desired estimate can be concluded by the same arguments as in case I.

*Case III:*  $N_1 \leq N_2$  and  $N_2 \gg N_3$ . We further distinguish between

$$\begin{aligned} IIIa_1 &: N_1 \gg N_3, N_1 \ll N_2, \\ IIIa_2 &: N_1 \gg N_3, N_1 \sim N_2, \\ IIIb &: N_1 \sim N_3, IIIc: N_1 \ll N_3. \end{aligned}$$



*Case IIIa<sub>1</sub>:*  $N_2 \gg N_1 \gg N_3$ . We will only obtain positive terms, if  $N_2 \sim N$  and, by resonance relation (10), we have  $\max\{|\tau + \xi^2|, |\tau_1 + \xi_1^2|, |\tau_2 + \xi_2^2|, |\tau_3 - \xi_3^2|\} \gtrsim N_1 N_2$ . This means

$$\begin{aligned} |M(\xi, \tau, \vec{\xi}, \vec{\tau})| &\lesssim \frac{1}{\langle \tau + \xi^2 \rangle^{1/2} \langle \tau_1 + \xi_1^2 \rangle^{1/2} \langle \tau_2 + \xi_2^2 \rangle^{1/2} \langle \tau_3 - \xi_3^2 \rangle^{1/2}} \\ &\lesssim N_1^{-1/16} N_2^{-1/16} M_4(\xi, \tau, \vec{\xi}, \vec{\tau}). \end{aligned} \quad (33)$$

Since  $N_2^{-1/16} \lesssim N_3^{-1/16}$ , we obtain the conclusion with arguments similar to case *IIb<sub>1,+</sub>*.

*Case IIIa<sub>2</sub>:*  $N_1 \sim N_2 \gg N_3$ . For a positive contribution, we need  $N_2 \gtrsim N$  and

$$\max\{|\tau + \xi^2|, |\tau_1 + \xi_1^2|, |\tau_2 + \xi_2^2|, |\tau_3 - \xi_3^2|\} \gtrsim N_1 N_2 \sim N_2^2 \gtrsim N N_2$$

by resonance relation (10). Hence  $|M(\xi, \tau, \vec{\xi}, \vec{\tau})| \lesssim \sum_{j=1}^3 M_j(\xi, \tau, \vec{\xi}, \vec{\tau})$ . Since  $N_2^{-1/8} \lesssim N_2^{-1/16} N_3^{-1/16}$ , we get the desired estimate by similar arguments as in *I*.

*Case IIIb:*  $N_2 \gg N_1 \sim N_3$ . We can assume that  $N_2 \sim N$  and

$$\max\{|\tau + \xi^2|, |\tau_1 + \xi_1^2|, |\tau_2 + \xi_2^2|, |\tau_3 - \xi_3^2|\} \gtrsim N_2 |\xi - \xi_2| \geq N_2^{1/2},$$

if  $|\xi - \xi_2| \geq N_2^{-1/2}$  – otherwise we use similar arguments to case *IIb<sub>1,-</sub>*. We obtain

$$\begin{aligned} |M(\xi, \tau, \vec{\xi}, \vec{\tau})| &\lesssim \frac{1}{\langle \tau + \xi^2 \rangle^{1/2} \langle \tau_1 + \xi_1^2 \rangle^{1/2} \langle \tau_2 + \xi_2^2 \rangle^{1/2} \langle \tau_3 - \xi_3^2 \rangle^{1/2}} \\ &\lesssim N_2^{-1/32} M_4(\xi, \tau, \vec{\xi}, \vec{\tau}) \end{aligned}$$

which implies the conclusion by arguing as in case *IIb<sub>1,+</sub>*.

*Case IIIc:*  $N_2 \gg N_3 \gg N_1$ . Here  $N_2 \sim N$  and

$$\max\{|\tau + \xi^2|, |\tau_1 + \xi_1^2|, |\tau_2 + \xi_2^2|, |\tau_3 - \xi_3^2|\} \gtrsim N_2 N_3.$$

Hence  $|M(\xi, \tau, \vec{\xi}, \vec{\tau})| \lesssim \sum_{j=1}^3 M_j(\xi, \tau, \vec{\xi}, \vec{\tau})$  and we obtain the desired estimate as argued in case *I*.  $\square$

**Theorem 4.6: Trilinear estimate for  $\mathcal{Y}^{\frac{1}{2}, -1}$ .** *Let  $T \in (0, 1]$ ,  $u_j \in \mathcal{S}(\mathbb{X} \times \mathbb{R})$  and  $\text{supp } u_j \subseteq \mathbb{X} \times [-T, T]$ ,  $j \in \{1, 2, 3\}$ . There is an  $\varepsilon > 0$  such that*

$$\|\mathcal{T}(u_1, u_2, u_3)\|_{\mathcal{Y}^{\frac{1}{2}, -1}} \lesssim T^\varepsilon \|u_1\|_{\mathcal{X}^{\frac{1}{2}, \frac{1}{2}}} \|u_2\|_{\mathcal{X}^{\frac{1}{2}, \frac{1}{2}}} \|u_3\|_{\mathcal{X}^{\frac{1}{2}, \frac{1}{2}}}. \quad (34)$$

*Proof.* By an application of the triangle inequality, we may assume that  $P_{N_j} u_j \geq 0$  and  $P_{N_j} u_j = \chi_T(t) P_{N_j} u_j$ . As in the main part of the previous proof, we will omit the case of small frequencies and focus on  $|\xi|$  of order  $N > 1$ . We have

$$\begin{aligned} &\|P_N \mathcal{T}(u_1, u_2, u_3)\|_{\mathcal{Y}^{\frac{1}{2}, -1}} \\ &\leq \sum_{N_1, N_2, N_3} \left\| \chi_N(\xi) \int_{\mathbb{R}^3_\tau} \int_{\mathbb{Y}^3_\xi} \tilde{M}(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_{N_j, u_j}(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L^2_\xi L^1_\tau}. \end{aligned} \quad (35)$$

We subdivide  $\tilde{M}$  into  $\tilde{M}_0, \tilde{M}_1, \tilde{M}_2, \tilde{M}_3, \tilde{M}_4$  and adapt the ideas of [13, Thm. 4.2] to our setting: For  $\gamma > 0$ ,  $\xi \in \mathbb{Y}$  and  $w: \mathbb{Y} \times \mathbb{R} \rightarrow \mathbb{C}$  such that  $w(\xi, \cdot) \in L^2(\mathbb{R})$ ,

Hölder's inequality and the transformation theorem lead to

$$\|\langle \tau \pm \xi^2 \rangle^{-\frac{1}{2}-\gamma} w(\xi, \tau)\|_{L_\tau^1} \lesssim_\gamma \|w(\xi, \tau)\|_{L_\tau^2}. \quad (36)$$

By definition of  $\tilde{M}_0$  and Young's convolution inequality, we get

$$\begin{aligned} & \left\| \chi_N(\xi) \int_{\mathbb{R}_\tau^3} \int_{\mathbb{Y}_\xi^3} \tilde{M}_0(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_{N_j, u_j}(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L_\xi^2 L_\tau^1} \\ & \lesssim \left\| \int_{\mathbb{Y}_\xi^3} \langle \xi \rangle^{-\frac{1}{2}+\delta} \prod_{j=1}^2 \left\| \frac{f_{N_j, u_j}(\xi_j, \tau_j)}{\langle \xi_j \rangle^{\frac{1}{2}} \langle \tau_j + \xi_j^2 \rangle^{\frac{\delta}{2}}} \right\|_{L_{\tau_j}^2} \cdot \left\| \frac{f_{N_3, u_3}(\xi_3, \tau_3)}{\langle \xi_3 \rangle^{\frac{1}{2}-3\delta} \langle \tau_3 - \xi_3^2 \rangle^{\frac{\delta}{2}}} \right\|_{L_{\tau_3}^2} d\vec{\xi} \right\|_{L_\xi^2}. \end{aligned} \quad (37)$$

Let

$$\begin{aligned} g_{N_j, u_j}(\xi, \tau) &:= \langle \tau + \xi^2 \rangle^{-\delta/2} f_{N_j, u_j}(\xi, \tau), \quad j \in \{1, 2\}, \\ g_{N_3, u_3}(\xi, \tau) &:= \langle \tau - \xi^2 \rangle^{-\delta/2} f_{N_3, u_3}(\xi, \tau). \end{aligned}$$

In the sequel, we choose  $\delta = \frac{1}{24}$ . Then, by Young's and Hölder's inequality,

$$\begin{aligned} (37) & \lesssim \left\| \int_{\mathbb{Y}_\xi^3} \langle \xi \rangle^{-3/8} \prod_{j=1}^2 \langle \xi_j \rangle^{-1/2} \cdot \langle \xi_3 \rangle^{-3/8} \prod_{j=1}^3 \|g_{N_j, u_j}(\xi_j, \tau_j)\|_{L_{\xi_j, \tau_j}^2} d\vec{\xi} \right\|_{L_\xi^2} \\ & \lesssim N_1^{-3/16} N_2^{-3/16} N_3^{-1/16} \prod_{j=1}^3 \|g_{N_j, u_j}(\xi_j, \tau_j)\|_{L_{\xi_j, \tau_j}^2} \\ & = N_1^{-3/16} N_2^{-3/16} N_3^{-1/16} \prod_{j=1}^2 \|P_{N_j} u_j\|_{X^{\frac{1}{2}, \frac{23}{48}}} \cdot \|P_{N_3} u_3\|_{X^{\frac{1}{2}, \frac{23}{48}, -}}. \end{aligned}$$

Hence

$$\begin{aligned} & \left\| \chi_N(\xi) \int_{\mathbb{R}_\tau^3} \int_{\mathbb{Y}_\xi^3} \tilde{M}_0(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_{N_j, u_j}(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L_\xi^2 L_\tau^1} \\ & \lesssim T^\varepsilon \|P_{N_1} u_1\|_{X^{\frac{5}{16}, \frac{1}{2}}} \|P_{N_2} u_2\|_{X^{\frac{5}{16}, \frac{1}{2}}} \|u_3\|_{X^{\frac{7}{16}, \frac{1}{2}, -}}. \end{aligned} \quad (38)$$

By definition of  $\tilde{M}_1$  and estimate (36), we have

$$\begin{aligned} & \left\| \chi_N(\xi) \int_{\mathbb{R}_\tau^3} \int_{\mathbb{Y}_\xi^3} \tilde{M}_1(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_{N_j, u_j}(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L_\xi^2 L_\tau^1} \\ & \lesssim \left\| \chi_N(\xi) \int_{\mathbb{R}_\tau^3} \int_{\mathbb{Y}_\xi^3} \langle \tau + \xi^2 \rangle^{1/8} M_1(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_{N_j, u_j}(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L_{\xi, \tau}^2} \\ & \lesssim \left\| \Gamma^{1/2} P_{N_1} u_1 \cdot P_{N_2} u_2 \cdot J_x^{1/2} P_{N_3} u_3 \right\|_{X^{0, -\frac{3}{8}}}. \end{aligned}$$

Applying (18) leads to

$$\begin{aligned} & \left\| \chi_N(\xi) \int_{\mathbb{R}_\tau^3} \int_{\mathbb{Y}_\xi^3} \tilde{M}_1(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_{N_j, u_j}(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L_\xi^2 L_\tau^1} \\ & \lesssim T^\varepsilon \|P_{N_1} u_1\|_{X^{\frac{3}{8}, \frac{1}{2}}} \|P_{N_2} u_2\|_{X^{\frac{3}{8}, \frac{1}{2}}} \|P_{N_3} u_3\|_{X^{\frac{1}{2}, \frac{1}{2}, -}}. \end{aligned} \quad (39)$$

By changing the first two factors, one can show analogously

$$\begin{aligned} & \left\| \chi_N(\xi) \int_{\mathbb{R}_\tau^3} \int_{\mathbb{Y}_\xi^3} \tilde{M}_2(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_{N_j, u_j}(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L_\xi^2 L_\tau^1} \\ & \lesssim T^\varepsilon \|P_{N_1} u_1\|_{X^{\frac{3}{8}, \frac{1}{2}}} \|P_{N_2} u_2\|_{X^{\frac{3}{8}, \frac{1}{2}}} \|P_{N_3} u_3\|_{X^{\frac{1}{2}, \frac{1}{2}, -}}. \end{aligned} \quad (40)$$

For  $\tilde{M}_3$ , we conclude by (36) that

$$\begin{aligned} & \left\| \chi_N(\xi) \int_{\mathbb{R}_\tau^3} \int_{\mathbb{Y}_\xi^3} \tilde{M}_3(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_{N_j, u_j}(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L_\xi^2 L_\tau^1} \\ & \lesssim \left\| \int_{\mathbb{R}_\tau^3} \int_{\mathbb{Y}_\xi^3} \langle \tau + \xi^2 \rangle^{-7/16} \langle \tau_3 - \xi_3^2 \rangle^{1/2} \langle \xi_3 \rangle^{1/2} \prod_{j=1}^3 \widehat{P_{N_j} u_j}(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L_{\xi, \tau}^2} \\ & \lesssim \|P_{N_1} u_1 \cdot P_{N_2} u_2 \cdot J_x^{1/2} \Gamma_-^{1/2} P_{N_3} u_3\|_{X^{0, -\frac{7}{16}}} \end{aligned}$$

and (19) implies

$$\begin{aligned} & \left\| \chi_N(\xi) \int_{\mathbb{R}_\tau^3} \int_{\mathbb{Y}_\xi^3} \tilde{M}_3(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_{N_j, u_j}(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L_\xi^2 L_\tau^1} \\ & \lesssim T^\varepsilon \|P_{N_1} u_1\|_{X^{\frac{3}{8}, \frac{1}{2}}} \|P_{N_2} u_2\|_{X^{\frac{3}{8}, \frac{1}{2}}} \|P_{N_3} u_3\|_{X^{\frac{1}{2}, \frac{1}{2}, -}}. \end{aligned} \quad (41)$$

By definition of  $\tilde{M}_4$  and estimate (36), we have

$$\begin{aligned} & \left\| \chi_N(\xi) \int_{\mathbb{R}_\tau^3} \int_{\mathbb{Y}_\xi^3} \tilde{M}_4(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_{N_j, u_j}(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L_\xi^2 L_\tau^1} \\ & \lesssim \left\| \int_{\mathbb{R}_\tau^3} \int_{\mathbb{Y}_\xi^3} \langle \tau + \xi^2 \rangle^{-1/2} M_4(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_{N_j, u_j}(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L_\xi^2 L_\tau^1} \\ & \lesssim \|J_x^{1/2} \Gamma_-^{1/16} P_{N_1} u_1 \cdot J_x^{1/2} \Gamma_-^{1/16} P_{N_2} u_2 \cdot J_x^{1/2} \Gamma_-^{1/16} P_{N_3} u_3\|_{X^{0, -\frac{7}{16}}}. \end{aligned}$$

From (20), we conclude

$$\begin{aligned} & \left\| \chi_N(\xi) \int_{\mathbb{R}_\tau^3} \int_{\mathbb{Y}_\xi^3} \tilde{M}_4(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_{N_j, u_j}(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L_\xi^2 L_\tau^1} \\ & \lesssim T^\varepsilon \|P_{N_1} u_1\|_{X^{\frac{1}{2}, \frac{1}{2}}} \|P_{N_2} u_2\|_{X^{\frac{1}{2}, \frac{1}{2}}} \|P_{N_3} u_3\|_{X^{\frac{1}{2}, \frac{1}{2}, -}}. \end{aligned} \quad (42)$$

Now, we show estimate (35). As in the proof of theorem 4.5, it suffices to consider  $\Omega'_\xi$  instead of  $\mathbb{Y}_\xi^3$ . We denote

$$\begin{aligned} & \sum_{N_1, N_2, N_3} \left\| \chi_N(\xi) \int_{\mathbb{R}_\tau^3} \int_{\Omega'_\xi} \tilde{M}(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_{N_j, u_j}(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L_\xi^2 L_\tau^1} \\ & \lesssim I + IIa + IIb_{1,+} + IIb_{1,-} + IIb_2 + IIIa_1 + IIIa_2 + IIIb + IIIc, \end{aligned}$$

where  $I, \dots, IIIc$  refer to the same cases as in the proof of theorem 4.5.

*Case I:*  $N_1 \leq N_2 \ll N_3$ . We have  $N \sim N_3$  and

$$\max\{|\tau + \xi^2|, |\tau_1 + \xi_1^2|, |\tau_2 + \xi_2^2|, |\tau_3 - \xi_3^2|\} \sim N N_3.$$

Since  $\tilde{M}(\xi, \tau, \vec{\xi}, \vec{\tau}) = \langle \tau + \xi^2 \rangle^{-1/2} M(\xi, \tau, \vec{\xi}, \vec{\tau})$ , it follows from the same arguments as in the proof of theorem 4.5 that

$$\begin{aligned} |\tilde{M}(\xi, \tau, \vec{\xi}, \vec{\tau})| &\lesssim \langle \tau + \xi^2 \rangle^{-1/2} \sum_{j=0}^3 M_j(\xi, \tau, \vec{\xi}, \vec{\tau}) \\ &= \langle \tau + \xi^2 \rangle^{-1/2} M_0(\xi, \tau, \vec{\xi}, \vec{\tau}) + \sum_{j=1}^3 \tilde{M}_j(\xi, \tau, \vec{\xi}, \vec{\tau}) \\ &\lesssim \sum_{j=0}^3 \tilde{M}_j(\xi, \tau, \vec{\xi}, \vec{\tau}), \end{aligned} \quad (43)$$

where we used resonance relation (10) in the last step as follows: For  $(\vec{\xi}, \vec{\tau}) \in A_0(\xi, \tau)$ , we have

$$\langle \tau + \xi^2 \rangle \gtrsim NN_3 \sim \langle \xi \rangle \langle \xi_3 \rangle.$$

Hence, for any  $\delta \in (0, \frac{1}{6})$ , it holds

$$\langle \tau + \xi^2 \rangle^{1/2} \gtrsim \langle \xi \rangle^{\frac{1}{2}-3\delta} \langle \xi_3 \rangle^{\frac{1}{2}-3\delta} \langle \tau_1 + \xi_1^2 \rangle^\delta \langle \tau_2 + \xi_2^2 \rangle^\delta \langle \tau_3 - \xi_3^2 \rangle^\delta. \quad (44)$$

Finally, by (38)-(41), we obtain

$$\begin{aligned} I &\lesssim T^\varepsilon \sum_{N_1, N_2} \sum_{N_3 \sim N} \|P_{N_1} u_1\|_{X^{\frac{3}{8}, \frac{1}{2}}} \|P_{N_2} u_2\|_{X^{\frac{3}{8}, \frac{1}{2}}} \|P_{N_3} u_3\|_{X^{\frac{1}{2}, \frac{1}{2}, -}} \\ &\lesssim T^\varepsilon \|u_1\|_{X^{\frac{1}{2}, \frac{1}{2}}} \|u_2\|_{X^{\frac{1}{2}, \frac{1}{2}}} \|u_3\|_{X^{\frac{1}{2}, \frac{1}{2}, -}}. \end{aligned}$$

*Case IIa:*  $N_1 \sim N_2 \sim N_3$ . Since  $N_1, N_2, N_3 \sim N$ , this follows from (38)-(42).

*Case IIb<sub>1, \pm</sub>:*  $N \sim N_1 \ll N_2 \sim N_3$ . Since  $|\tilde{M}(\xi, \tau, \vec{\xi}, \vec{\tau})| = \langle \tau + \xi^2 \rangle^{-1/2} |M(\xi, \tau, \vec{\xi}, \vec{\tau})|$ , estimate (28) implies

$$|\tilde{M}(\xi, \tau, \vec{\xi}, \vec{\tau})| \lesssim \frac{1}{\langle \tau + \xi^2 \rangle \langle \tau_1 + \xi_1^2 \rangle^{1/2} \langle \tau_2 + \xi_2^2 \rangle^{1/2} \langle \tau_3 - \xi_3^2 \rangle^{1/2}}.$$

First, consider  $\vec{\xi} \in \Omega'_{\xi, +}$  which means  $|\xi - \xi_1| \geq N_3^{-1/2}$ . As in (29), we have

$$|\tilde{M}(\xi, \tau, \vec{\xi}, \vec{\tau})| \lesssim N_1^{-1/64} N_3^{-1/64} \langle \tau + \xi^2 \rangle^{-1/2} M_4(\xi, \tau, \vec{\xi}, \vec{\tau}). \quad (45)$$

An application of (36) and Strichartz estimates (7), (6) together with Hölder's inequality provides

$$\begin{aligned} &\left\| \chi_N(\xi) \int_{\mathbb{R}_\tau^3} \int_{\Omega'_{\xi, +}} \langle \tau + \xi^2 \rangle^{-1/2} M_4(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_{N_j, u_j}(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L_\xi^2 L_\tau^1} \\ &\lesssim \|J_x^{1/2} \Gamma^{1/16} P_{N_1} u_1 \cdot J_x^{1/2} \Gamma^{1/16} P_{N_2} u_2 \cdot J_x^{1/2} \Gamma_-^{1/16} P_{N_3} u_3\|_{X^{0, -\frac{13}{32}}} \\ &\lesssim \|J_x^{1/2} \Gamma^{1/16} P_{N_1} u_1\|_{L_{t,x}^4} \|J_x^{1/2} \Gamma^{1/16} P_{N_2} u_2\|_{L_{t,x}^4} \|J_x^{1/2} \Gamma_-^{1/16} P_{N_3} u_3\|_{L_{t,x}^4} \\ &\lesssim T^\varepsilon \|P_{N_1} u_1\|_{X^{\frac{1}{2}, \frac{1}{2}}} \|P_{N_2} u_2\|_{X^{\frac{1}{2}, \frac{1}{2}}} \|P_{N_3} u_3\|_{X^{\frac{1}{2}, \frac{1}{2}, -}} \end{aligned}$$

and we obtain the desired estimate by (45), (21) and (23).

Now, let  $\vec{\xi} \in \Omega'_{\xi,-}$  which means  $|\xi - \xi_1| < N_3^{1/2}$ . Estimate (36) implies

$$\begin{aligned} & \left\| \chi_N(\xi) \int_{\mathbb{R}_\tau^3} \int_{\Omega'_{\xi,-}} \tilde{M}(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_{N_j, u_j}(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L_\xi^2 L_\tau^1} \\ & \lesssim \left\| \chi_N(\xi) \int_{\mathbb{R}_\tau^3} \int_{\Omega'_{\xi,-}} \langle \tau + \xi^2 \rangle^{-1/2} M(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_{N_j, u_j}(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L_\xi^2 L_\tau^1} \\ & \lesssim \left\| \chi_N(\xi) \int_{\mathbb{R}_\tau^3} \int_{\Omega'_{\xi,-}} \langle \tau + \xi^2 \rangle^{1/16} M(\xi, \tau, \vec{\xi}, \vec{\tau}) \prod_{j=1}^3 f_{N_j, u_j}(\xi_j, \tau_j) d\vec{\xi} d\vec{\tau} \right\|_{L_{\xi, \tau}^2}. \end{aligned}$$

Finally, we can argue as done in the proof of theorem 4.5: Note that estimate (7) used in (31) is also valid for  $b = -\frac{7}{16}$  instead of  $b = -\frac{1}{2}$ . Hence, we can ignore the extra term  $\langle \tau + \xi^2 \rangle^{1/16}$ , here.

*Case IIb<sub>2</sub>*:  $N_1 \ll N$  and  $N_1 \ll N_2 \sim N_3$ . We may assume that

$$\max\{|\tau + \xi^2|, |\tau_1 + \xi_1^2|, |\tau_2 + \xi_2^2|, |\tau_3 - \xi_3^2|\} \gtrsim NN_3$$

which provides the conclusion by arguing as in case *I*.

*Case IIIa<sub>1</sub>*:  $N_2 \gg N_1 \gg N_3$ . We have  $N_2 \sim N$  and

$$\max\{|\tau + \xi^2|, |\tau_1 + \xi_1^2|, |\tau_2 + \xi_2^2|, |\tau_3 - \xi_3^2|\} \gtrsim N_2 N_1.$$

This means

$$\begin{aligned} |\tilde{M}(\xi, \tau, \vec{\xi}, \vec{\tau})| &= \langle \tau + \xi^2 \rangle^{-1/2} |M(\xi, \tau, \vec{\xi}, \vec{\tau})| \\ &\lesssim \langle \tau + \xi^2 \rangle^{-1/2} N_1^{-1/16} N_2^{-1/16} M_4(\xi, \tau, \vec{\xi}, \vec{\tau}) \end{aligned}$$

by (33) and we get the desired estimate using  $N_2^{-1/16} \leq N_3^{-1/16}$  as in case *IIb<sub>1,+</sub>*.

*Case IIIa<sub>2</sub>*:  $N_1 \sim N_2 \gg N_3$ . We need  $N_2 \gtrsim N$  and

$$\max\{|\tau + \xi^2|, |\tau_1 + \xi_1^2|, |\tau_2 + \xi_2^2|, |\tau_3 - \xi_3^2|\} \gtrsim N_2^2.$$

Hence  $|\tilde{M}(\xi, \tau, \vec{\xi}, \vec{\tau})| \lesssim \sum_{j=0}^3 \tilde{M}_j(\xi, \tau, \vec{\xi}, \vec{\tau})$  by (43). Since  $N_3^{-1/8} \lesssim N_2^{-1/16} N_3^{-1/16}$ , the conclusion follows from (38)-(41).

*Case IIIb*:  $N_2 \gg N_1 \sim N_3$ . Here,  $N_2 \sim N$  and

$$\max\{|\tau + \xi^2|, |\tau_1 + \xi_1^2|, |\tau_2 + \xi_2^2|, |\tau_3 - \xi_3^2|\} \gtrsim N_2^{1/2},$$

if  $|\xi - \xi_1| \geq N_2^{-1/2}$  – otherwise we use the ideas of *IIb<sub>1,-</sub>*. Hence

$$|\tilde{M}(\xi, \tau, \vec{\xi}, \vec{\tau})| \lesssim N_2^{-1/32} \langle \tau + \xi^2 \rangle^{-1/2} M_4(\xi, \tau, \vec{\xi}, \vec{\tau})$$

which implies the desired estimate as in case *IIb<sub>1,+</sub>*.

*Case IIIc*:  $N_2 \gg N_3 \gg N_1$ . We may assume  $N_2 \sim N$  and

$$\max\{|\tau + \xi^2|, |\tau_1 + \xi_1^2|, |\tau_2 + \xi_2^2|, |\tau_3 - \xi_3^2|\} \gtrsim N_2 N_3.$$

Therefore

$$|\tilde{M}(\xi, \tau, \vec{\xi}, \vec{\tau})| \lesssim \langle \tau + \xi^2 \rangle^{-1/2} M_0(\xi, \tau, \vec{\xi}, \vec{\tau}) + \sum_{j=1}^3 M_j(\xi, \tau, \vec{\xi}, \vec{\tau}) \lesssim \sum_{j=0}^3 \tilde{M}_j(\xi, \tau, \vec{\xi}, \vec{\tau}),$$

where the last step can be seen by using  $\langle \tau + \xi^2 \rangle \gtrsim N_2 N_3 \sim N N_3$  for  $(\vec{\xi}, \vec{\tau}) \in A_0(\xi, \tau)$  and a calculation as in (44). We obtain the conclusion by arguments similar to case I using  $N_2^{-1/8} \lesssim N_2^{-1/16} N_3^{-1/16}$ .  $\square$

**Corollary 4.7: Trilinear estimate for  $s \geq \frac{1}{2}$ .** *Let  $s \geq \frac{1}{2}$ ,  $\delta > 0$ ,  $T \in (0, 1]$ ,  $u_j \in \mathcal{S}(\mathbb{X} \times \mathbb{R})$  such that  $\text{supp } u_j \subseteq \mathbb{X} \times [-T, T]$ ,  $j \in \{1, 2, 3\}$ . Then, for some  $\varepsilon > 0$ ,*

$$\|\mathcal{T}(u_1, u_2, \overline{u_3})\|_{\mathfrak{X}^{s, -\frac{1}{2}} \cap \mathcal{Y}^{s, -1}} \lesssim T^\varepsilon \sum_{k=1}^3 \|u_k\|_{\mathfrak{X}^{s, \frac{1}{2}}} \prod_{\substack{j=1 \\ j \neq k}}^3 \|u_j\|_{\mathfrak{X}^{\frac{1}{2}, \frac{1}{2}}}.$$

*Proof.* As argued before, we focus on frequencies  $|\xi|$  of order  $N > 1$ . Since  $\langle \xi \rangle^{s-\frac{1}{2}} \lesssim \sum_{k=1}^3 \langle \xi_k \rangle^{s-\frac{1}{2}}$ , we have

$$\begin{aligned} \|P_N \mathcal{T}(u_1, u_2, \overline{u_3})\|_{\mathfrak{X}^{s, -\frac{1}{2}}} &\lesssim \left\| P_N \mathcal{T}(J_x^{s-\frac{1}{2}} u_1, u_2, \overline{u_3}) \right\|_{X^{\frac{1}{2}, -\frac{1}{2}}} \\ &\quad + \left\| P_N \mathcal{T}(u_1, J_x^{s-\frac{1}{2}} u_2, \overline{u_3}) \right\|_{X^{\frac{1}{2}, -\frac{1}{2}}} \\ &\quad + \left\| P_N \mathcal{T}(u_1, u_2, J_x^{s-\frac{1}{2}} \partial_x \overline{u_3}) \right\|_{X^{\frac{1}{2}, -\frac{1}{2}}}. \end{aligned}$$

Estimate (25) and  $\|\overline{u}\|_{X^{s, b, -}} = \|u\|_{X^{s, b}}$  imply

$$\|\mathcal{T}(u_1, u_2, \overline{u_3})\|_{\mathfrak{X}^{s, -\frac{1}{2}}} \lesssim T^\varepsilon \sum_{k=1}^3 \|u_k\|_{\mathfrak{X}^{s, \frac{1}{2}}} \prod_{\substack{j=1 \\ j \neq k}}^3 \|u_j\|_{\mathfrak{X}^{\frac{1}{2}, \frac{1}{2}}}.$$

Replacing (25) by (34) provides the same upper bound for  $\|\mathcal{T}(u_1, u_2, \overline{u_3})\|_{\mathcal{Y}^{s, -1}}$ .  $\square$

## 5. MULTILINEAR ESTIMATE

In this section, we consider the polynomial terms  $\mathcal{Q}(v)$  and  $|v|^{2k}v$ ,  $k \in \mathbb{N}_0$ . The absence of derivatives in these terms leads to a less technical proof.

**Theorem 5.1.** *Let  $s \geq \frac{1}{2}$ ,  $\delta > 0$ ,  $k \in \mathbb{N}_0$ ,  $T \in (0, 1]$ ,  $u_j \in \mathcal{S}(\mathbb{X} \times \mathbb{R})$  satisfying  $\text{supp } u_j \subseteq \mathbb{X} \times [-T, T]$ ,  $j \in \mathbb{N}_{\leq k+1}$ . There is an  $\varepsilon > 0$  such that*

$$\left\| \prod_{j=1}^{k+1} u_j \right\|_{\mathfrak{X}^{s, -\frac{3}{8}-\delta} \cap \mathcal{Y}^{s, -1}} \lesssim T^\varepsilon \sum_{l=1}^{k+1} \|u_l\|_{\mathfrak{X}^{s, \frac{1}{2}, \pm}} \prod_{\substack{j=1 \\ j \neq l}}^{k+1} \|u_j\|_{\mathfrak{X}^{\frac{1}{2}, \frac{1}{2}, \pm}} \quad (46)$$

and in particular

$$\left\| \mathcal{Q}(u_1, \overline{u_2}, u_3, \overline{u_4}, u_5) \right\|_{\mathfrak{X}^{s, -\frac{3}{8}-\delta} \cap \mathcal{Y}^{s, -1}} \lesssim T^\varepsilon \sum_{l=1}^5 \|u_l\|_{\mathfrak{X}^{s, \frac{1}{2}}} \prod_{\substack{j=1 \\ j \neq l}}^5 \|u_j\|_{\mathfrak{X}^{\frac{1}{2}, \frac{1}{2}}}. \quad (47)$$

*Proof.* By triangle inequality, we may assume  $P_{N_j} u_j \geq 0$  and  $P_{N_j} u_j = \chi_T(t) P_{N_j} u_j$ . Hence, estimate (47) is a direct consequence of (46). According to (3), we have  $X^{s, -\frac{3}{8}-\delta} \hookrightarrow Y^{s, -1}$  for  $\delta \in (0, \frac{1}{8})$ . Therefore, it suffices to handle the  $\mathfrak{X}^{s, -\frac{3}{8}-\delta}$ -norm.

For  $k = 0$ , we can conclude estimate (46) from  $X^{s, \frac{1}{2}} \hookrightarrow X^{s, -\frac{3}{8}-\delta}$  for  $\delta > 0$  and  $X^{s, -\frac{3}{8}-\delta} \hookrightarrow Y^{s, -1}$  for  $\delta \in (0, \frac{1}{8})$ , compare (3).

Now, let  $k \geq 1$ . As before, we focus on  $N > 1$ . Applying  $\langle \xi \rangle^s \lesssim \sum_{l=1}^{k+1} \langle \xi_l \rangle^s$  leads to

$$\left\| P_N \prod_{j=1}^{k+1} u_j \right\|_{X^{s, -\frac{3}{8}-\delta}} \lesssim \sum_{l=1}^{k+1} \left\| P_N \left( J_x^s u_l \prod_{\substack{j=1 \\ j \neq l}}^{k+1} u_j \right) \right\|_{X^{0, -\frac{3}{8}-\delta}}.$$

Let  $\varepsilon \in (0, \frac{1}{2})$ ,  $l \in \mathbb{N}_{\leq k+1}$  and  $B_l := \{(N_1, \dots, N_{k+1}) \in \mathcal{D}_1^{k+1} : N_j \ll N_l \ \forall j \neq l\}$  (with an implicit constant greater than  $4k$ ). We decompose

$$\begin{aligned} & \left\| P_N \left( J_x^s u_l \prod_{\substack{j=1 \\ j \neq l}}^{k+1} u_j \right) \right\|_{X^{0, -\frac{3}{8}-\delta}} \\ & \leq \sum_{(N_1, \dots, N_{k+1}) \in B_l} \left\| \chi_N(\xi) \langle \tau + \xi^2 \rangle^{-\frac{3}{8}-\delta} \left( P_{N_k} J_x^s u_l \prod_{\substack{j=1 \\ j \neq l}}^{k+1} P_{N_j} u_j \right)^\wedge(\xi, \tau) \right\|_{L_{\xi, \tau}^2} \\ & \quad + \sum_{(N_1, \dots, N_{k+1}) \in B_l^c} \left\| \chi_N(\xi) \langle \tau + \xi^2 \rangle^{-\frac{3}{8}-\delta} \left( P_{N_k} J_x^s u_l \prod_{\substack{j=1 \\ j \neq l}}^{k+1} P_{N_j} u_j \right)^\wedge(\xi, \tau) \right\|_{L_{\xi, \tau}^2} \\ & =: I + II. \end{aligned}$$

*Case I:* We need  $N_l \sim N$  for a positive contribution. From Strichartz estimate (7), Hölder's inequality and estimate (4), we conclude

$$\begin{aligned} I & \lesssim \sum_{N_l \sim N} \sum_{N_j, j \neq l} \left\| P_{N_l} J_x^s u_l \prod_{\substack{j=1 \\ j \neq l}}^{k+1} P_{N_j} u_j \right\|_{L_{t,x}^{4/3}} \\ & \lesssim \sum_{N_l \sim N} \|P_{N_l} u_l\|_{X^{s, 0, \pm}} \prod_{j=1}^{k+1} \sum_{\substack{N_j \\ j \neq l}} \|P_{N_j} u_j\|_{X^{\frac{1}{2} - \frac{1}{4k}, \frac{1}{2}, \pm}} \lesssim T^\varepsilon \|u_l\|_{\mathfrak{X}^{s, \frac{1}{2}, \pm}} \prod_{\substack{j=1 \\ j \neq l}}^{k+1} \|u_j\|_{\mathfrak{X}^{\frac{1}{2}, \frac{1}{2}, \pm}}. \end{aligned}$$

*Case II:* For  $(N_1, \dots, N_{k+1}) \in B_l^c$ , there is a  $j_l^* \in \mathbb{N}_{\leq k+1} \setminus \{l\}$  such that  $N_l \lesssim N_{j_l^*}$ . This means

$$\begin{aligned} II & \lesssim \sum_{N_j, j \neq l} \sum_{N_l \lesssim N_{j_l^*}} \left\| P_{N_l} J_x^s u_l \prod_{\substack{j=1 \\ j \neq l}}^{k+1} P_{N_j} u_j \right\|_{L_{t,x}^{4/3}} \\ & \lesssim \sum_{N_j, j \neq l} \sum_{N_l \lesssim N_{j_l^*}} \|P_{N_{j_l^*}} u_{j_l^*}\|_{X^{\frac{1}{2} - \frac{1}{8k}, \frac{1}{2}, \pm}} \|P_{N_l} u_l\|_{X^{s - \frac{1}{8k}, 0, \pm}} \prod_{\substack{j=1 \\ j \neq l, j_l^*}}^{k+1} \|u_j\|_{\mathfrak{X}^{\frac{1}{2}, \frac{1}{2}, \pm}} \\ & \lesssim T^\varepsilon \|u_l\|_{\mathfrak{X}^{s, \frac{1}{2}, \pm}} \prod_{\substack{j=1 \\ j \neq l}}^{k+1} \|u_j\|_{\mathfrak{X}^{\frac{1}{2}, \frac{1}{2}, \pm}} \end{aligned}$$

using the dual Strichartz estimate (7) in the first step, Hölder's inequality and estimate (4) in the second step.  $\square$

## 6. LOCAL WELL-POSEDNESS

By the standard contraction mapping principle, we obtain the following local well-posedness result for the gauge equivalent problem:

**Theorem 6.1.** *Let  $s \geq \frac{1}{2}$ ,  $k \in \mathbb{N}_0$ ,  $r > 0$  and  $B_r := \{v_0 \in B_{2,\infty}^s(\mathbb{X}) : \|v_0\|_{B_{2,\infty}^s(\mathbb{X})} < r\}$ . For any  $v_0 \in B_r$ , there is a  $T = T(r) > 0$  such that the Cauchy problem (2) has a unique solution  $v \in \mathcal{Z}_T^s$ . The flow map*

$$F: B_r \rightarrow \mathcal{C}([-T, T], B_{2,\infty}^s(\mathbb{X})), \quad v_0 \mapsto v$$

*is Lipschitz continuous.*

We can conclude local well-posedness for equation (1) (i.e. prove theorem 1.1) by the same strategy as in Herr [13]: The Gauge transformation is a locally bilipschitz homeomorphism on  $\mathcal{C}([-T, T], B_{2,\infty}^s)$  which can be shown by an application of Sobolev's multiplication theorem for Besov spaces:

$$\|f_1 f_2\|_{B_{2,\infty}^s} \lesssim \|f_1\|_{B_{2,\infty}^{s_1}} \|f_2\|_{B_{2,\infty}^{s_2}} \quad (48)$$

for  $f_1 \in B_{2,\infty}^{s_1}$ ,  $f_2 \in B_{2,\infty}^{s_2}$ ,  $s \geq 0$ ,  $s_1, s_2 \geq s$ ,  $s_1 + s_2 - s > \frac{1}{2}$ . A proof for  $H^s$  instead of  $B_{2,\infty}^s$  can be found for example in [14, Corollary 1.1.12]. With trivial modifications, one can show (48) in a similar way by localizing frequencies on  $N_1, N_2$  and considering the cases  $N_1 \sim N_2$ ,  $N_1 \ll N_2$ . We obtain the statement of theorem 1.1 by establishing  $M_{s,T} := G^{-1}(\mathcal{Z}_T^s)$  in the non-periodic setting and  $M_{s,T} := \mathcal{G}^{-1}(\mathcal{Z}_T^s)$  in the periodic setting.

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